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Problem 1 (10 points)

A bead of mass, \( m \), is constrained to a frictionless “V-shaped” track, and this track rotates about the \( z \)-axis, as shown in the figure. The equation of the track is \( z = k \rho \), where \( \rho \) is the distance perpendicular to the \( z \)-axis and \( k \) is a positive constant. The V-shaped track rotates about \( \hat{z} \) with constant angular velocity, \( \omega \).

(a) [4 points] Taking the perpendicular distance from the \( z \)-axis, \( \rho \), as the variable of interest, write the equation of motion for \( \rho \).

(b) [2 points] Are there any equilibrium points?

(c) [4 points] Show that the equation of motion from part (a) reduces to the equation of motion for a frictionless inclined plane if the track is not being rotated.
Problem 2 (10 points)

A solid cylinder (radius $R$, mass $M$ with uniform density $\rho$, length $L$, volume $\pi R^2 L$) is attached via a spring (massless, spring constant $k$) to a wall as shown in the figure. The horizontal surface is rough enough that the cylinder can only roll (no slipping).

(a) [4 points] Perform the integral $I = \int r^2dm$ over the cylinder to show that the moment of inertia of the cylinder for rotations about its symmetry axis is $I = \frac{1}{2}MR^2$.

(b) [4 points] Letting the generalized coordinate be $x$, the position of the center of the cylinder with respect to its equilibrium position, what is the equation of motion?

(c) [2 points] What is the angular frequency, $\omega$, for small oscillations about the equilibrium position?
Problem 3 (10 points)

A block of mass $m$ is projected upward along an inclined plane that makes an angle $\theta$ with the horizontal plane as shown in the figure below. At the block-incline interface both the coefficients of static and kinetic friction have the same value $\mu$. The initial speed of the block at the bottom of the incline is $v_0$.

(a) [2 points] Determine, in terms of given quantities, the maximum distance $D$ that the block moves up along the inclined plane and the time, $t_{up}$, it takes to reach that highest point.

(b) [2 points] Next, the block slides down the ramp to its starting point. What is the time of descent, $t_{down}$, of the block to the bottom of the ramp?

(c) [2 points] What is the largest value of the angle $\theta$, in terms of given quantities, such that the block stops and stays at its highest point?

(d) [2 points] Now imagine that the block and the inclined plane are frictionless (that is, $\mu=0$). What is the total up-and-down travel time for the block in this frictionless case?

(e) [2 points] What is the relationship between $\mu$ and $\theta$ if the total travel times are the same for the frictional and the frictionless motions?
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Problem 4 (10 points)

Two semi-infinite conducting sheets are grounded and connected at a right angle as seen in the figure. One sheet lies in the $\hat{x}\hat{z}$ plane and the other sheet lies in the $\hat{y}\hat{z}$ plane. A point charge, $q$, is placed at the coordinate $(x, y, z) = (a, b, 0)$.

(a) [2 points] Determine the locations of the image charges required to calculate the electric potential, $V(x, y, z)$, for $x, y > 0$.

(b) [3 points] Determine the electric potential, $V(x, y, z)$, for $x, y > 0$.

(c) [3 points] Calculate the work required to bring the charge to $(x, y, z) = (a, b, 0)$ from a point that is infinitely far away from the sheets.

(d) [2 points] Calculate the charge densities, $\sigma(x, z)$ and $\sigma(y, z)$, that are induced on the sheet in the $\hat{x}\hat{z}$ plane and on the sheet in the $\hat{y}\hat{z}$ plane.
Problem 5 (10 points)

A spherically symmetric charge distribution exists within a sphere of radius $R$. The electric field inside the sphere points radially outward, and is of the form $\mathbf{E} = [Ar + Br^2]\hat{r}$.

(a) [2 points] Calculate the charge density, $\rho(r)$.

(b) [2 points] Calculate the electric field outside the sphere ($r > R$).

(c) [2 points] Calculate the electric potential, $V(r)$, for a point outside the sphere ($r > R$).

(d) [2 points] Calculate the electric potential, $V(r)$, for a point inside the sphere ($r < R$).

(e) [2 points] How much work is required to assemble the charge distribution?

Hint: In spherical coordinates the divergence of a vector function can be expressed as:

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (A_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}. $$
**Problem 6 (10 points)**

An infinite conducting slab is centered about the \( \hat{x}\hat{y} \)-plane and is shown in the upper figure to the right. The slab has a total thickness of \( 2d \). A current density \( \mathbf{J} = (kz^2)\hat{x} \) flows throughout the slab.

(a) [1 point] Determine the direction that the magnetic field points for \( z < 0 \) and \( z > 0 \).

(b) [3 points] Determine the magnitude of the magnetic field, \( \mathbf{B}(z) \), at some point inside the slab (\( |z| < d \)).

(c) [3 points] Determine the magnitude of the magnetic field, \( \mathbf{B}(z) \), at some point outside the slab (\( |z| > d \)).

A loop of wire with resistance \( R \) and capacitance \( C \) is placed a height \( h \) above the top surface of the conducting slab. As shown in the lower figure on the right, this loop is in the \( \hat{z}\hat{x} \)-plane. The loop has a length \( L \) and width \( w \). At \( t = 0 \), a switch in the loop is closed and a complete circuit is formed. Simultaneously, the current density in the slab begins to increase linearly in time and is given by \( \mathbf{J} = (ktz^2/T)\hat{x} \), where \( T \) is a time constant.

(d) [2 points] Determine the direction of current flow in the loop of wire.

(e) [1 point] Calculate the current induced in the loop of wire as a function of time.
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Problem 7 (10 points)

A system has three energy levels at $\epsilon=0$, $\epsilon=k_B T_a$, and $\epsilon=k_B T_b$, where $k_B$ is the Boltzmann constant, $T_a=300$ K and $T_b=600$ K. The degeneracies of the levels are 1, 3, and 5, respectively. To receive full credit, provide numerical answers for all parts.

(a) [3 points] Calculate the single-particle partition function at a temperature of 300K.

(b) [2 points] Calculate the relative populations of the energy levels at 300K.

(c) [2 points] Calculate the average energy per particle at 300K.

(d) [3 points] At what temperature is the population of the energy level at $k_B T_b$ equal to the population of the energy level at $k_B T_a$?
Problem 8 (10 points)

A heat engine runs in Joule’s cycle, which consists of two constant pressure ($P$) processes and two constant entropy ($S$) processes, as shown in the diagram. Assume that the working material is a monatomic ideal gas.

(a) [2 points] Make a qualitative drawing of the cycle in the pressure-volume ($P$-$V$) diagram. Label the isobaric (constant pressure) steps. Label the isentropic (constant entropy) steps.

(b) [1 point] If the engine is to be run to produce work, will the cycle be clockwise or counterclockwise in the $P$-$V$ diagram?

(c) [1 point] Which step in the $P$-$V$ diagram has the heat coming into the engine? Give a reason.

(d) [1 point] Which step in the $P$-$V$ diagram has the heat going out of the engine? Give a reason.

(e) [5 points] What is the efficiency of this heat engine in terms of the variables $P_1$ and $P_2$ only?

The following information might be useful.
For a constant-entropy process, $PV^\gamma = \text{constant}$, $TV^{\gamma^{-1}} = \text{constant}$, and $TP^{(1-\gamma)/\gamma} = \text{constant}$, with $\gamma = 5/3$ for monatomic ideal gas.
Problem 9 (10 points)

A particle of mass $m_0$ at rest can decay into two particles of rest masses $m_1$ and $m_2$ only if the initial mass $m_0$ is greater than the sum of final masses, i.e. if the mass excess $\Delta = m_0 - m_1 - m_2$ is positive.

(a) [3 points] Derive the relativistic expressions for the kinetic energies of the two particles formed in the decay process with the mass $m_0$ initially at rest.

(b) [2 points] Show that the relativistic expressions for the kinetic energies of the two particles can be written as,

$$T_i = \left[1 - \frac{m_i}{m_0} - \frac{\Delta}{2m_0}\right] \Delta c^2, \quad (i = 1, 2).$$

(c) [2 points] Verify explicitly that the sum of the kinetic energies of the two particles equals $\Delta c^2$.

(d) [3 points] A charged $\pi$ meson of rest energy 139.6 MeV decays into a $\mu$ meson of rest energy 105.7 MeV and a neutrino with zero rest mass. Calculate the kinetic energies of the $\mu$ meson and the neutrino.
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Problem 10 (10 points)

A particle of mass $m$ is confined to move freely inside a two-dimensional square box, with impenetrable sides each of length $L$. The sides are parallel to the $\hat{x}$ and $\hat{y}$ axes with $0 \leq x \leq L$ and $0 \leq y \leq L$.

(a) [4 points] Derive the energy eigenvalues and the corresponding eigenfunctions of this particle.

(b) [3 points] What is the degeneracy of the lowest two allowed energy states?

(c) [3 points] Now, a small perturbation $V = \beta xy$ is introduced, where $\beta$ is a constant. Calculate the energy shift of the ground state through first order perturbation theory.
Problem 11 (10 points)

Consider the single-particle state of an electron.

(a) [5 points] For a corresponding spin operator $\hat{S}_x + \hat{S}_y + \hat{S}_z$, what are its possible eigenvalues? What is the normalized eigenvector (also called eigenspinor) corresponding to the smallest eigenvalue?

(b) [5 points] Initially the electron is in an eigenstate corresponding to the smallest eigenvalue of the above spin operator given in (a). What is the probability that a measurement of $\hat{S}_x$ gives the outcome of $+\hbar/2$? (Hint: first calculate the corresponding eigenvector of $\hat{S}_x$.)

Note the Pauli matrices are written as

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
**Problem 12** (10 points)

A quantum rotator is governed by the Hamiltonian

\[ \hat{H} = \frac{\hat{L}_x^2}{2I_1} + \frac{\hat{L}_y^2}{2I_2}, \]

where \( L_x, L_y, \) and \( L_z \) are the three components of orbital angular momentum, and \( I_1 \neq I_2 \) are the moments of inertia and are constants.

(a) [3 points] What are the energy eigenvalues of this rotator?

(b) [1 point] What are the corresponding normalized eigenfunctions (obtained without any new calculations)? You need to briefly explain your answer.

(c) [6 points] When this rotator is subjected to a perturbation \( \hat{H}' = E \sin(2\theta) \), where \( E \) is a small constant, calculate the first-order energy corrections for the states with the azimuthal quantum number \( l = 1 \).

Note the following spherical harmonics:

\[ Y_0^0 = \left( \frac{1}{4\pi} \right)^{1/2}, \quad Y_1^0 = \left( \frac{3}{4\pi} \right)^{1/2} \cos \theta, \quad Y_1^\pm = \mp \left( \frac{3}{8\pi} \right)^{1/2} \sin \theta e^{\pm i\phi}. \]