MAT 2010 Final Exam Solution W23

1. $f(x) = 3 + \sqrt{x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{3 + \sqrt{x+h} - 3 - \sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} * \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim_{h\to 0} \frac{x+h-x}{h[\sqrt{x+h} + \sqrt{x}]} = \lim_{h\to 0} \frac{h}{h[\sqrt{x+h} + \sqrt{x}]}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

2. (a) $\lim_{x\to 0} \frac{\cos x - 1}{e^x - 1} \to \lim_{x\to 0} \frac{(\cos x - 1)'}{(e^x - 1)'}$ (using L'Hopital's rule) $= \lim_{x\to 0} \frac{-\sin(x)}{e^x} \to \frac{-\sin 0}{e^0} = \frac{0}{1} = 0$

(b) $\lim_{x\to -1} \frac{x^2 - 2x}{2x^2 - 5x + 3} = \frac{(-1)^2 - 2(-1)}{2(-1)^2 - 5(-1) + 3} = \frac{1+2}{2+5+3} = \frac{3}{10}$

(c) $\lim_{x\to\infty} \frac{3x+2}{\sqrt{5x^2 - x}} \div \frac{x}{\sqrt{x^2}} = \lim_{x\to\infty} \frac{\frac{3x+2}{x}}{\frac{\sqrt{5x^2-x}}{\sqrt{x^2}}} = \lim_{x\to\infty} \frac{3 + \frac{2}{x}}{\sqrt{5 - \frac{1}{x}}} = \frac{3 + \lim_{x\to\infty}\frac{2}{x}}{\sqrt{5 - \lim_{x\to\infty}\frac{1}{x}}} = \frac{3+0}{\sqrt{5-0}} = \frac{3}{\sqrt{5}}$ or $\frac{3\sqrt{5}}{5}$

3. (a) $f(x) = \sin(2x) * \ln(x+1)$

$$f'(x) = [\sin(2x) * \frac{1}{x+1}] + [\ln(x+1) * 2\cos(2x)]$$

$$f'(x) = \frac{\sin(2x)}{x+1} + 2\cos(2x)\ln(x+1)$$

(b) $g(x) = \arctan\left(\frac{1}{x}\right)$

$$g'(x) = \frac{1}{1 + \left(\frac{1}{x}\right)^2} * \left(-\frac{1}{x^2}\right) = -\frac{1}{x^2+1}$$

4. (a) $\int [\frac{3}{x} - 5\sec(x)\tan(x) + \sec(x)^2]dx = 3\ln|x| - 5\sec(x) + \tan(x) + C$

(b) $\int_1^4 \left(\frac{5}{\sqrt{x}} - 3\right) dx = \int_1^4 \left(5x^{-\frac{1}{2}}\right) dx = \left[5\frac{x^{\frac{1}{2}}}{\frac{1}{2}} - 3x\right]_1^4 = 10\sqrt{x} - 3x|_1^4 = 8 - 7 = 1$

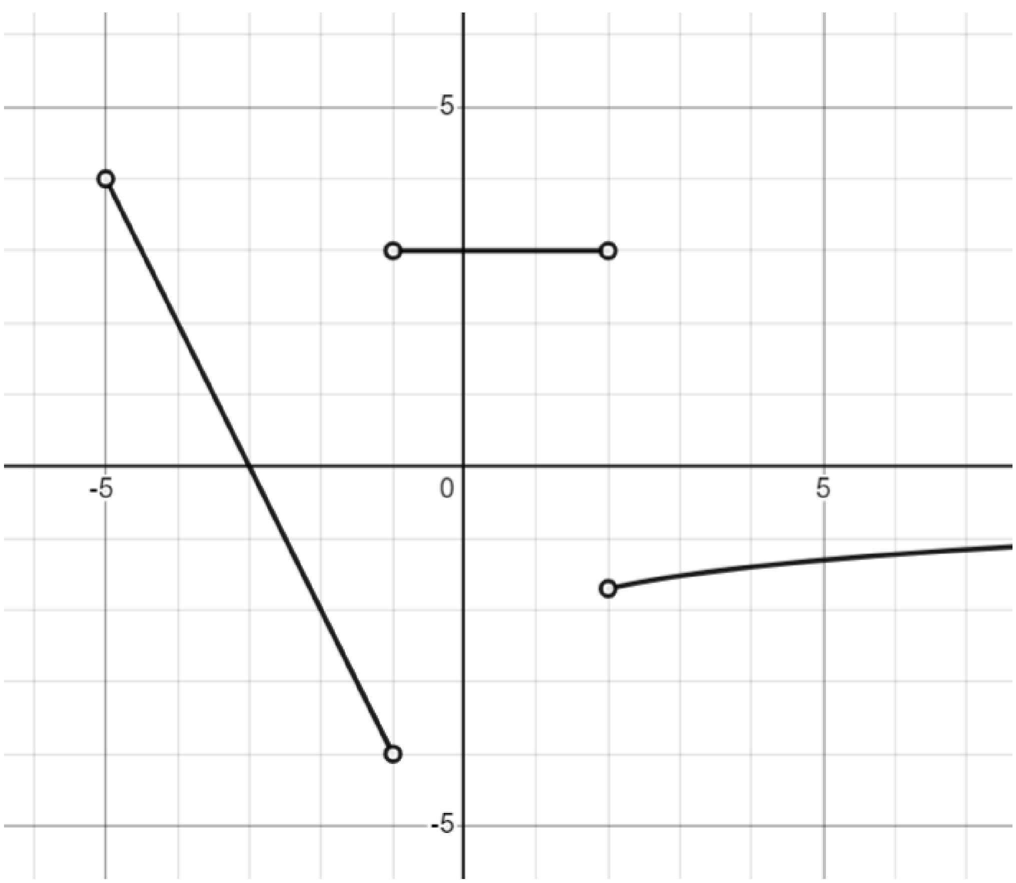
5. $f(x) = \frac{2x^2+3x}{x^2-3}$

$$f'(x) = \frac{(x^2-3)(4x+3)-(2x^2+3x)(2x)}{(x^2-3)^2} = \frac{4x^2-12x+3x^2-9-4x^2-6x^2}{(x^2-3)^2}$$

$$= \frac{-3x^2-12x-9}{(x^2-3)^2} = \frac{-3(x^2+4x+3)}{(x^2-3)^2}$$

For Horizontal Tangent: $f'(x) = 0 \rightarrow -3(x^2+4x+3) = 0 \rightarrow (x^2+4x+3) = 0$
$\rightarrow (x+3)(x+1) = 0 \rightarrow x = -3, x = -1$

6.



7. $f(x) = \frac{x}{x+1}$ $\quad [1,9]$

$a = 1, \quad b = 9, \quad n = 4$

$\Delta x = \frac{b-a}{n} = \frac{8}{4} = 2$

$\boldsymbol{x}$	$\boldsymbol{x_0 = 1}$	$\boldsymbol{x_1 = 3}$	$\boldsymbol{x_2 = 5}$	$\boldsymbol{x_3 = 7}$	$\boldsymbol{x_4 = 9}$
$\boldsymbol{f(x)}$	1/2	3/4	5/6	7/8	9/10

$$Area = A \approx \Delta * [f(1) + f(3) + f(5) + f(7)]$$

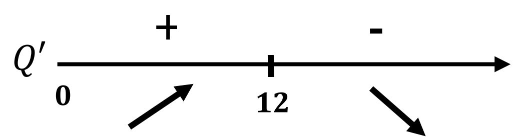
$$= 2\left[\frac{1}{2} + \frac{3}{4} + \frac{5}{6} + \frac{7}{8}\right] = 2\left[\frac{12 + 18 + 20 + 21}{24}\right] = 2\left[\frac{71}{24}\right] = \frac{71}{12}$$

8. $x + 3y = 54 \ (constraint)$ $\quad$ (Discard 0, we need positive numbers)

$Q = xy^2 \quad (maximize)$

$x = 54 - 3y$

1st Derivative Test



local max @ $y = 12$

$Q = (54 - 3y)y^2 = 54y^2 - 3y^3$

$Q' = 54(y) - 3(3y^3) = 108y - 9y^2$

$Q' = 0 \ \rightarrow 108y - 9y^2 = 0$

$y(12 - y) = 0$ $\quad \rightarrow x = 54 - 3(12) = 54 - 36 = 18$

$\rightarrow y = 0 \ or \ y = 12$

Thus, the two numbers are 18 and 12.

9. (a) $g)(1) = \int_{-5}^{1} f(t)dt = A_1 + A_2 = \frac{1}{2} * 2 * 4 + \frac{1}{2} * 4 * 3 = -4 + 6 = 2$

(b) g is increasing on $(-3,1)$

(c) g is concave down on $(0,3)$

(d) g has no local maximum value at $x = 1$

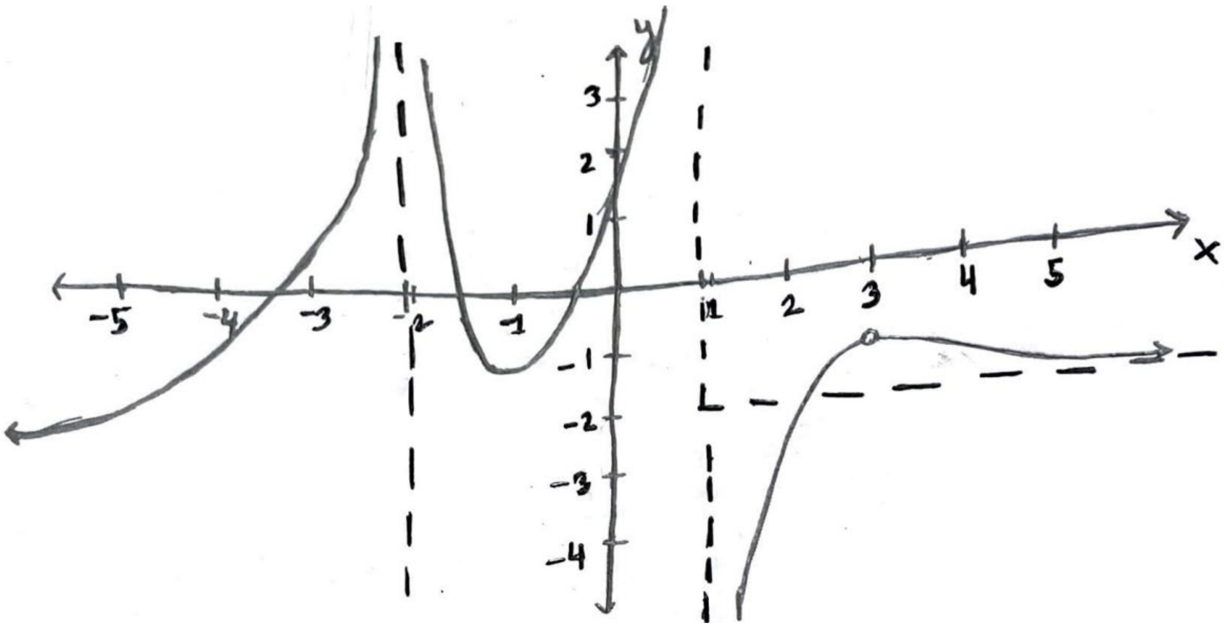
(e)

i. $g'(3) = f(3) = 2$

ii. $g''(3) = f'(3) = DNE$

iii. $g''(-1) = f'(-1) = 1$

10.



This is one possible solution.

11. $\frac{dV}{dt} = \frac{100cm^3}{s}$ $\frac{dr}{dt} =?$ when $r = \frac{20}{2} = 10$cm

$V = \frac{4}{3}\pi r^3 \rightarrow \frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right) \rightarrow \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$

$100 = 4\pi(10)^2 * \frac{dr}{dt} \rightarrow 100 = 4\pi(100) * \frac{dr}{dt} \rightarrow \frac{dr}{dt} = \frac{1}{4\pi}$

The radius of the balloon is increasing at a rate of $\frac{1}{4\pi}$ cm/s

12. $f'(x) = \frac{9(x^2-4)}{(x^2+4)^2} = \frac{9(x+2)(x-2)}{(x^2+4)(x^2+4)}$

$f''(x) = -\frac{18x(x^2-12)}{(x^2+4)^3} = 0 \quad \rightarrow \quad x = 0 \ \ or \ x^2 - 12 = 0$ (finding the zeros of $f''(x)$)

$\rightarrow x^2 = 12$

$\rightarrow x \pm \sqrt{12}$

$= \pm 2\sqrt{3}$

f is increasing on $(-\infty, -2) \cup (0,2)$

f is decreasing on $(-2,0) \cup (2, \infty)$

f has a local maximum at $x = -2$

f has a local minimum at $x = 2$

f is concave up on $(-\infty, -2\sqrt{3}) \cup (0, 2\sqrt{3})$

f is concave down on $(-2\sqrt{3}, 0) \cup (2\sqrt{3}, \infty)$

Points of Inflection: $x = -2\sqrt{3}, 0, 2\sqrt{3}$