FINAL EXAMINATION, MAT 2010
April 28, 2023

INSTRUCTIONS
Write your solutions in a blue book. To receive full credit you must show *all* work. You are allowed to use an *approved* graphing calculator unless otherwise indicated. Simplify your answers when possible, but use the precise value rather than an approximation when you have a choice. (Example: If the actual answer is π, then write π, not 3.14.) The 12 problems are worth a total of 150 points. The time limit is 2 hours [120 minutes].

Use of cell phones and other electronic devices is not allowed. They should be turned off and put away.

#1. (10 points) Use the **definition** of the derivative to differentiate the following function.

$$f(x) = 3 + \sqrt{x}$$

(No credit will be awarded for calculating the derivative *without* using the definition of the derivative.)

#2. (7 points each) Find the exact value of each of the following limits. Write "∞," "$-\infty$," or "does not exist" if appropriate. It is particularly important to show detailed algebraic work in finding each limit rather than plugging in numbers to estimate the limit.

(a) $\displaystyle\lim_{x \to 0} \frac{\cos x - 1}{e^x - 1}$

(b) $\displaystyle\lim_{x \to 2} \frac{x^2 - 2x}{2x^2 - 5x + 2}$

(c) $\displaystyle\lim_{x \to \infty} \frac{3x + 2}{\sqrt{5x^2 - x}}$

#3. (7 points each) Differentiate the following functions.

(a) $f(x) = \sin(2x)\,\ln(x+1)$

(b) $g(x) = \arctan\left(\frac{1}{x}\right)$

#4. Evaluate.

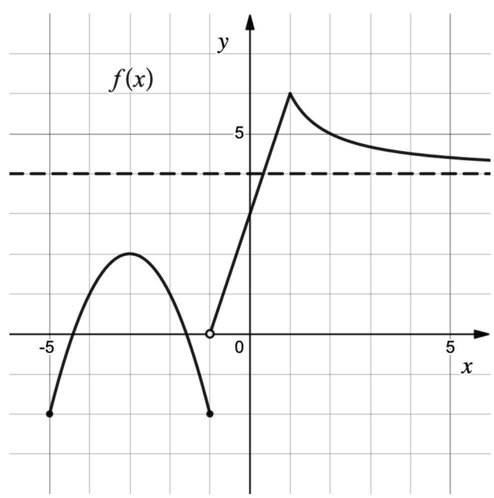
(a) (7 points) $\displaystyle\int \left[\frac{3}{x} - 5\sec x \tan x + \sec^2 x\right] dx$

(b) (8 points) $\displaystyle\int_1^4 \left[\frac{5}{\sqrt{x}} - 3\right] dx$

#5. (10 points) Find all x values at which the following function has horizontal tangents,

$$f(x) = \frac{2x^2 + 3x}{x^2 - 3}$$

#6. (10 points) The graph of a function $f(x)$ is shown below.

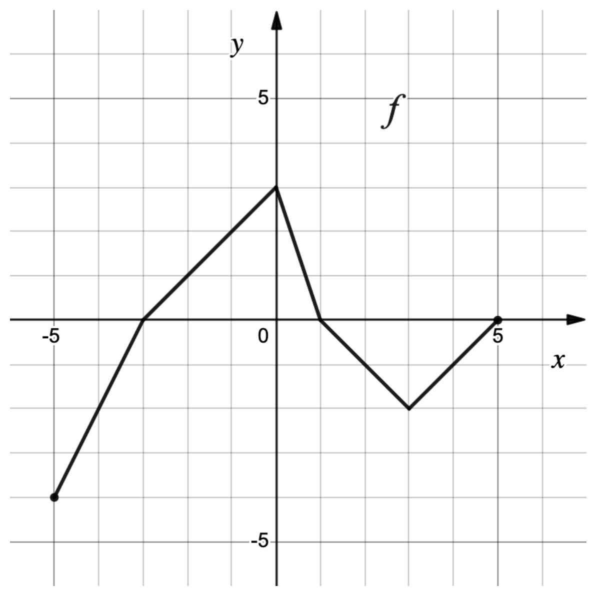


Sketch the graph of the derivative $f'(x)$ showing clearly where $f'(x)$ is positive and negative, and intervals where $f'(x)$ increases or decreases.

#7. (10 points) Using the left-endpoint Riemann sum with 4 equal subintervals estimate the area bounded above by the curve $f(x) = \dfrac{x}{x+1}$ and below by the x-axis on the interval $[1, 9]$. Give exact answer.

#8. (10 points) Let x and y be positive numbers such that $x + 3y = 54$. Find the values of x and y that will give the largest possible value of the quantity $Q = xy^2$.

#9. (10 points) The graph of a function f is shown below.



Define a new function $$g(x) = \int_{-5}^{x} f(t)\, dt, \qquad -5 \le x \le 5.$$

(a) Find $g(1)$.

(b) Give the subinterval(s) of the interval $[-5, 5]$ where $g(x)$ is increasing.

(c) Give the subinterval(s) of the interval $[-5, 5]$ where $g(x)$ is concave down.

(d) Give value(s) of x in the interval $(-5, 5)$ where $g(x)$ has local maximum value(s).

(e) Find (i) $g'(3)$ (ii) $g''(3)$ (iii) $g''(-1)$

#10. (10 points) Sketch the graph of a function f such that all the following conditions are satisfied:

- $\lim_{x \to 3} f(x)$ exists, but $f(3)$ is undefined;
- $\lim_{x \to -2} f(x) = \infty$, $\lim_{x \to 1^-} f(x) = \infty$, and $\lim_{x \to 1^+} f(x) = -\infty$;
- $\lim_{x \to \infty} f(x) = -2$.

#11. (10 points) Air is being pumped into a spherical balloon so that its volume V increases at a rate of $100\,\text{cm}^3/\text{s}$. How fast is the radius r of the balloon increasing when the diameter is 20 cm? [The volume of a spherical balloon is given by $V = \frac{4}{3}\pi r^3$.]

#12. (20 points) Given the following information for a function $f(x)$.

(i) $f(x)$ is defined for all real numbers

(ii) $f'(x) = \dfrac{9\,(x^2-4)}{(x^2+4)^2}$

(iii) $f''(x) = -\dfrac{18\,x\,(x^2-12)}{(x^2+4)^3}$

(iv) $f(0) = 0$

(v) $\lim_{x \to \infty} f(x) = 0$

(vi) $\lim_{x \to -\infty} f(x) = 0$

(vii) The function has a local maximum value of 2.25.

Find

(a) All intervals on which $f(x)$ is increasing or decreasing.

(b) x-value(s) of all local (relative) extrema.

(c) All intervals where $f(x)$ is concave up or concave down.

(d) x-value(s) of all inflection points.

(e) Sketch the graph of $f(x)$. Label all asymptotes, local extrema, and inflection points.