FINAL EXAMINATION, MAT 2010

April 29, 2022

INSTRUCTIONS

Write your solutions in a blue book. To receive full credit you must show *all* work. You are allowed to use an *approved* graphing calculator unless otherwise indicated. Simplify your answers when possible, but use the precise value rather than an approximation when you have a choice. (Example: If the actual answer is π, then write π, not 3.14.) The 12 problems are worth a total of 150 points. The time limit is 2 hours [120 minutes].

Cell phones are strictly prohibited!

1. (10 points) Use the **definition** of the derivative to differentiate the function

$$f(x) = x^2 - 3x + 2.$$

(No credit will be awarded for calculating the derivative *without* using the definition of the derivative.)

2. (7 points each) Find the exact value of each of the following limits. Write "∞," "$-\infty$," or "does not exist" if appropriate. It is particularly important to show your work on this problem. Numerical approximations do not constitute an acceptable solution.

   (a) $\lim\limits_{x \to 1} \dfrac{x-1}{x^2 - \tan(\pi x)}$

   (b) $\lim\limits_{x \to \infty} \sqrt{\dfrac{8x^2+3}{2x^2+x}}$

3. (7 points each) Differentiate the following functions. Simplify your answer.

   (a) $f(x) = x^{3/2}\left(e^x + e^3\right)$

   (b) $g(x) = \sqrt{\ln\left(2x+1\right)}$

4. Evaluate.

(a) (7 points) $\int \left[\frac{7}{1+x^2} - \frac{5}{\sqrt{1-x^2}}\right] dx$

(b) (8 points) $\int_0^{\pi/3} \Big(1 \,+\, 2\sec\theta\tan\theta\Big)\, d\theta$

(Give an exact answer in (b). Do not convert to decimals.)

5. (7 points) Sketch the graph of a function g that has **all** of the following properties:

- g is defined at all real numbers,
- the graph has horizontal asymptotes given by $y = -2$ and $y = 2$,
- $\lim_{x \to 3} g(x)$ exists but g is not continuous at $x = 3$,
- g is continuous but not differentiable at $x = -1$.

6. (14 points) For the curve

$$y = \frac{(x+1)^2}{1+x^2}$$

(a) Find $\frac{dy}{dx}$.

(b) Find the equation of the tangent line to the curve at $x = 0$.

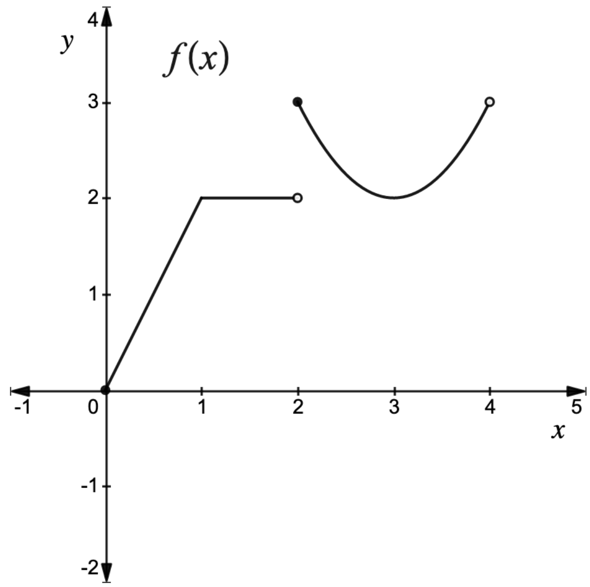
(c) Find all values of x where the graph of the curve has horizontal tangents.

7. (10 points) The total surface area S of a right circular cylinder with height h and base of radius r is given by the formula

$$S = 2\pi r^2 + 2\pi r h$$

A certain cylinder's radius is increasing at the rate of $\frac{1}{2}$ in/sec and its height is decreasing at the rate of 1 in/sec. How fast is the surface area S of the cylinder changing when its radius is 2 in and its height is 3 in? Give the exact answer with proper units.

8. (10 points) The graph of a function f is given below. Sketch the graph of the derivative function $f'(x)$.



9. (14 points) The following function represents the number of milligrams of a drug in a patient's bloodstream, t hours after the drug is administered.

$$f(t) = 30\,e^{-0.23t}$$

(a) Find the total change in the amount from $t = 0$ until $t = 2$. Give answer correct to two decimal places. Include proper units.

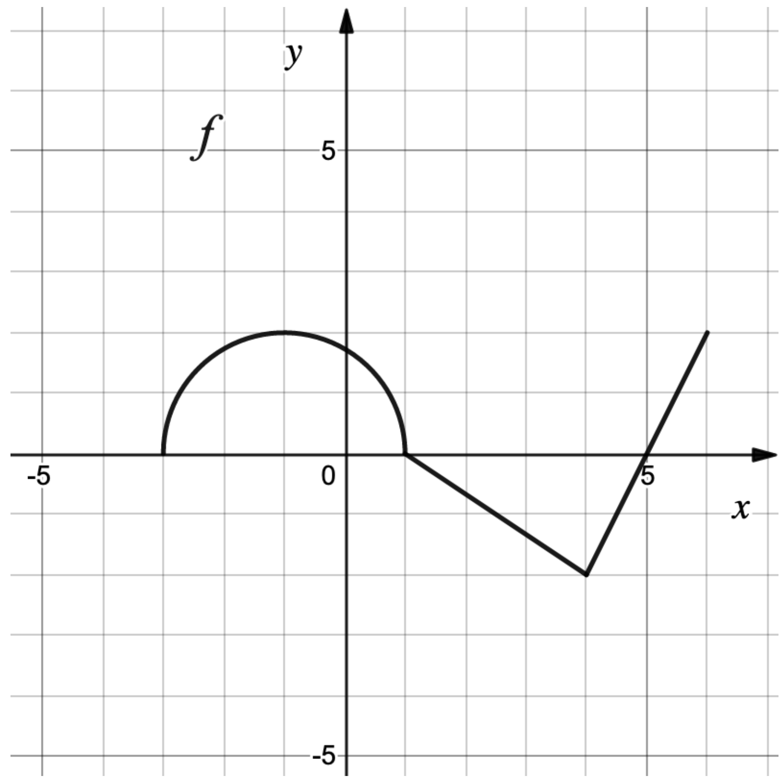
(b) Find the average rate of change in the amount from $t = 0$ until $t = 2$. Give the answer correct to two decimal places. Include proper units.

(c) Find the instantaneous rate of change in the amount at $t = 1$. Give the answer correct to two decimal places. Include proper units.

10. (12 points) Find the absolute minimum and absolute maximum values for the function

$$f(x) = \cos x - \sin^2 x \qquad \text{on} \qquad [0, \pi].$$

11. The graph of a function f consists of a semicircle and two line segments as shown.



Define a new function $g(x) = \int_{-3}^{x} f(t)dt$.

(a) (4 points) Find $g(5)$.

(b) (2 points) Find all values of x on the open interval $(-3, 4)$ at which g has a local (relative) maximum.

(c) (2 points) Find all sub-intervals in $(-3, 4)$ where g is concave up.

(d) (2 points) Find the values of x on the open interval $(-3, 4)$ at which g has an inflection point.

12. (20 points) Sketch the graph of a single function $f(x)$ that satisfies all of the following conditions. Find critical numbers for $f'(x)$ and $f''(x)$ and show sign charts for both functions. Indicate the intervals where $f(x)$ is increasing, decreasing, concave up and concave down. Give x-coordinates of all local maxima, local minima, and inflection points and label them. Show asymptotes using dashed lines.

(i) $f(x)$ is defined and continuous for all real numbers except at $x = 0$ and $x = 2$

(ii) $f'(x) = -\dfrac{1}{x^2 - 2x}$

(iii) $f''(x) = \dfrac{2(x-1)}{x^2(x-2)^2}$

(iv) $\lim_{x \to -\infty} f(x) = 0$

(v) $\lim_{x \to \infty} f(x) = 0$

(vi) $\lim_{x \to 0} f(x) = -\infty$

(vii) $\lim_{x \to 2} f(x) = \infty$

(viii) $f(1) = 0$