MAT 2010 Winter'21

1. (a) $\lim_{x\longrightarrow\infty} \frac{\sqrt{2x^2-7x}}{5x+1} = \lim_{x\longrightarrow\infty} \frac{\sqrt{2-\frac{7}{x}}}{5+\frac{1}{x}} = \frac{\sqrt{2-0}}{5+0} = \boxed{\frac{\sqrt{2}}{5}}$.

   (b) $\lim_{x\longrightarrow 0} \frac{\arctan x}{\arcsin x} =$ (L'Hospital) $\lim_{x\longrightarrow 0} \frac{\frac{1}{1+x^2}}{\frac{1}{\sqrt{1-x^2}}} = \frac{\frac{1}{1+0}}{\frac{1}{\sqrt{1-0}}} = \frac{1}{1} = \boxed{1}$.

2. (a) $f'(x) = (3^x.\sec x)' = (3^x)'.\sec x + 3^x.(\sec x)' = \boxed{3^x ln3.\sec x + 3^x.\sec x.\tan x}$.

   (b) $g'(x) = \left(\frac{3x^2-5x}{\tan x}\right)' = \frac{(3x^2-5x)'.\tan x - (3x^2-5x).(\tan x)'}{(\tan x)^2}$

   $= \boxed{\frac{(6x-5).\tan x - (3x^2-5x).\sec^2 x}{(\tan x)^2}}$.

   (c) $h'(x) = (\ln(e^{3x}+e^{-3x}))' = ((e^{3x}+e^{-3x}))'.\frac{1}{e^{3x}+e^{-3x}}$

   $= \boxed{\frac{3e^{3x}-3e^{-3x}}{e^{3x}+e^{-3x}}}$.

3. (a) $\int \left(\frac{\sqrt{x^5}}{x^3} - e^x - \cos 3\right)dx = \int \left(\frac{1}{\sqrt{x}} - e^x - \cos 3\right)dx = \boxed{2\sqrt{x} - e^x - \cos 3.x + C}$.

   (b) $\int_0^{\frac{\pi}{4}} (1+2\cos\theta)d\theta = \theta\Big|_0^{\frac{\pi}{4}} + 2\sin x\Big|_0^{\frac{\pi}{4}} = \boxed{\frac{\pi}{4}+\sqrt{2}}$.

4. $f(x) = \sqrt{x+5}$

$$f'(x) = \lim_{h \longrightarrow 0} \frac{\sqrt{x+h+5} - \sqrt{x+5}}{h}$$

$$= \lim_{h \longrightarrow 0} \frac{\sqrt{x+h+5} - \sqrt{x+5}}{h} . \frac{\sqrt{x+h+5} + \sqrt{x+5}}{\sqrt{x+h+5} + \sqrt{x+5}}$$

$$= \lim_{h \longrightarrow 0} \frac{x+h+5-x-5}{h(\sqrt{x+h+5} + \sqrt{x+5})}$$

$$= \lim_{h \longrightarrow 0} \frac{1}{\sqrt{x+h+5} + \sqrt{x+5}}$$

$$= \boxed{\frac{1}{2\sqrt{x+5}}}.$$

5. For the curve $x^2y - 2y^3 + x + 3y = -7$

   (a) $\dfrac{d(x^2y - 2y^3 + x + 3y)}{dx} = \dfrac{d(-7)}{dx}$

   $$2xy + x^2\frac{dy}{dx} - 6y^2\frac{dy}{dx} + 1 + 3\frac{dy}{dx} = 0$$

   $$\frac{dy}{dx}(x^2 - 6y^2 + 3) = -2xy - 1$$

   $$\boxed{\frac{dy}{dx} = \frac{-2xy - 1}{x^2 - 6y^2 + 3}}$$

   (b) Tangent line at $(1, 2)$:

   $$y - y_0 = \frac{dy}{dx}(x_0, y_0).(x - x_0)$$

   $$y - 2 = \frac{dy}{dx}(1, 2).(x - 1)$$

   $$y - 2 = \frac{1}{4}.(x - 1)$$

   $$\boxed{y = \frac{1}{4}x + \frac{7}{4}}.$$

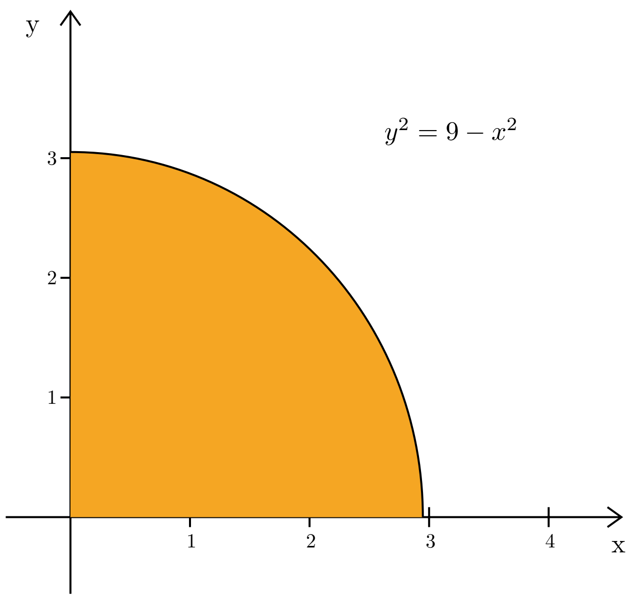
6. $\int_0^3 \left(|x-1| + \sqrt{9-x^2}\right)dx = \int_0^3 |x-1|dx + \int_0^3 \sqrt{9-x^2}dx$

$= \int_0^1 (1-x)dx + \int_1^3 (x-1)dx + \int_0^3 \sqrt{9-x^2}dx$

$= (x - \frac{x^2}{2})\Big|_0^1 + (\frac{x^2}{2} - x)\Big|_1^3 + \frac{1}{4}\pi(3)^2 \qquad (3)$

$= \frac{1}{2} + 2 + \frac{9}{4}\pi = \boxed{\frac{3}{2} + \frac{9}{4}\pi}.$

For the third equality (3), since $\sqrt{9-x^2} \geq 0$, we can interpret this integral as the area under the curve $y = \sqrt{9-x^2}$ from 0 to 3. Since $y^2 = 9 - x^2$, we get $x^2 + y^2 = 9$, which shows that the graph of $\sqrt{9-x^2}$ is the quarter-circle with radius 3. Therefore $\int_0^3 \sqrt{9-x^2} = \frac{1}{4}\pi(3)^2 = \frac{9}{4}\pi$.



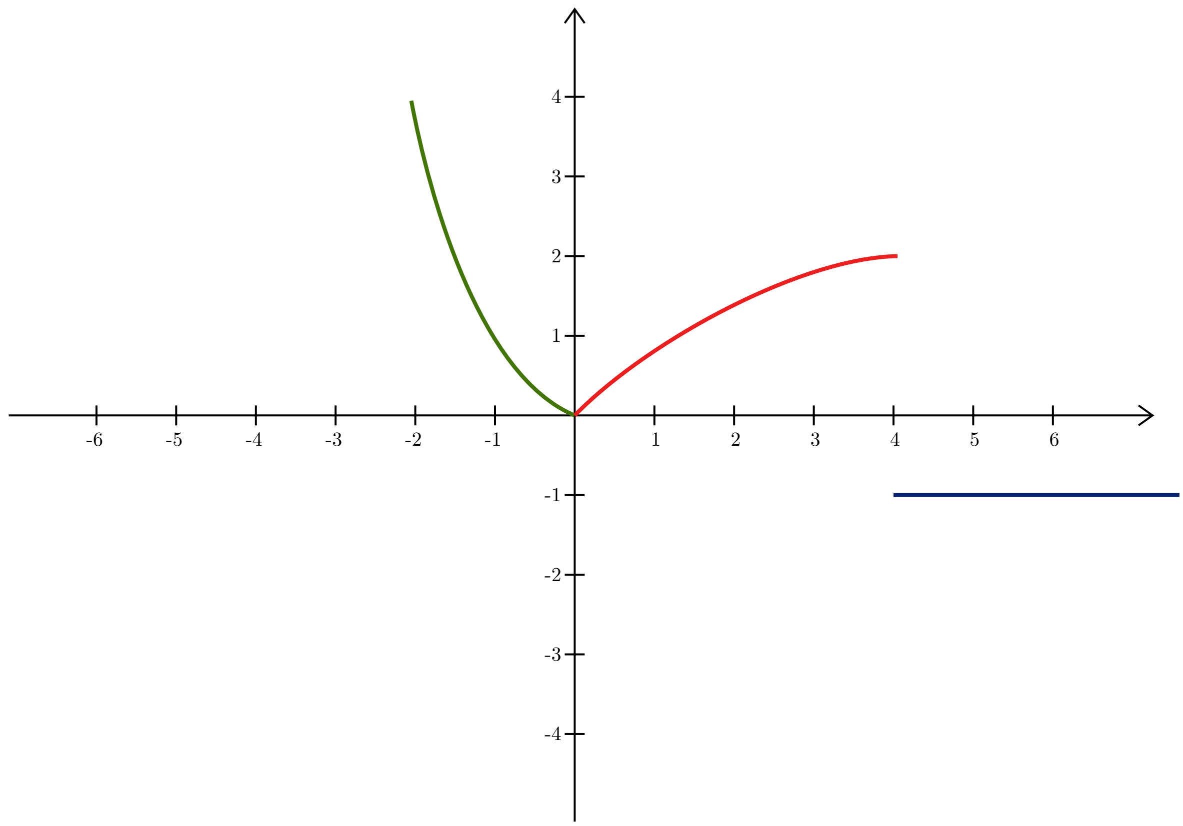
7. (a) f has a constant slope $\boxed{[6, 10]}$.

(b) f is increasing $\boxed{[3, 6]}$.

(c) local min: $\boxed{x = 3, x \in [6, 10]}$.

(d) f is concave up $\boxed{\left[\frac{1}{2}, \frac{9}{2}\right]}$.

8. (a)



(b) f is not continuous at $\boxed{x = 4}$.

(c) f is not differentialble at $\boxed{x = 0, x = 4}$.

9. $x^2 y = 13,500 \longrightarrow y = \dfrac{13,500}{x^2}$

The amount material $SA = x^2 + 4xy$

We need to solve the problem (P): min $f(x) = x^2 + \dfrac{5,400}{x}$

$f'(x) = 2x - \dfrac{5,400}{x^2} = 0 \longrightarrow \boxed{x = 30}, \boxed{y = 15}.$

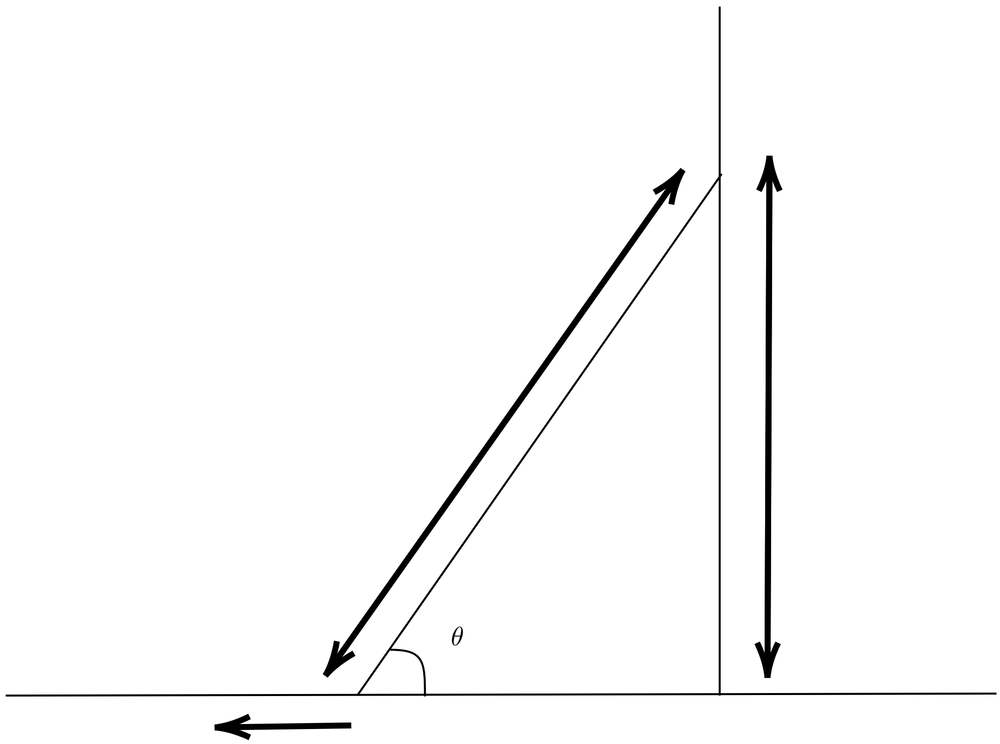
x	0		30		∞
$f'(x)$	-		0		+
$f(x)$		↘		↗	

10. We have $\dfrac{dy}{dt} = -1.5$ ft/min, $\quad \theta = \dfrac{2\pi}{9}$ rad, $\quad \sin\theta = \dfrac{y}{8}$,

Take derivatives for both sides:

$$\frac{d\theta}{dt}\cos\theta = \frac{dy}{8dt} = \frac{-1.5}{8}$$

$$\frac{d\theta}{dt} = \frac{-1.5}{8.\cos\dfrac{2\pi}{9}} = \boxed{-0.24}.$$



11. $a(t) = 6t + 4$, $\quad v(0) = 6$ cm/s, $\quad s(0) = 9$ cm,

$$v(t) = v(0) + \int_0^t a(s)ds = 6 + \int_0^t (6s+4)ds = 3t^2 + 4t + 6$$

$$s(t) = s(0) + \int_0^t v(s)ds = 9 + \int_0^t (3s^2 + 4s + 6)ds = \boxed{t^3 + 2t^2 + 6t + 9}.$$

12. (a) $f'(x) = \dfrac{3(x^2-1)}{x^4} = 0 \longrightarrow x = -1, 1.$

x	$-\infty$		-1		0		1		∞
$f'(x)$		$+$	0	-	0	-	0	$+$	
$f(x)$		↗		↘		↘		↗	

(b) f increase: $(-\infty, -1), (1, \infty)$

f decrease: $(-1, 0), (0, 1)$

f has a local minima: $x = 1$

f has a local maxima: $x = -1$.

(c) $f"(x) = \dfrac{6(2-x^2)}{x^5} = 0 \longrightarrow x = \sqrt{2}, -\sqrt{2}.$

x	$-\infty$		-$\sqrt{2}$		0		$\sqrt{2}$		∞
$f"(x)$		$+$	0	-	0	$+$	0	-	

(d) Inflection points: $x = \sqrt{2}$ and $x = -\sqrt{2}$

Concave up: $(-\infty, \sqrt{2}), (0, \sqrt{2})$

Concave down: $(-\sqrt{2}, 0), (\sqrt{2}, \infty)$.

(e) Horizontal asymptote: $y = 2$

Vertical asymptote: $x = 0$.

(f)