MAT 2010 Winter'22

1. $f(x) = x^2 - 3x + 2$, Use the definition of the derivation to differentiate $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - 3(x+h) + 2 - (x^2 - 3x + 2)}{h} =$$

$$\lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 3x - 3h + 2 - x^2 + 3x - 2}{h} = \lim_{h \to 0} \frac{h^2 + 2xh - 3h}{h} = \lim_{h \to 0} \frac{h(h + 2x - 3)}{h} =$$

$$\lim_{h \to 0} (h + 2x - 3) = \boxed{2x - 3}$$

2. (a) $\lim_{x \to 1} \dfrac{x - 1}{x^2 - \tan(\pi x)} = \dfrac{1 - 1}{1^2 - \tan(\pi.1)} = \dfrac{0}{1 - 0} = \boxed{0}$.

   (b) $\lim_{x \to \infty} \sqrt{\dfrac{8x^2 + 3}{2x^2 + x}} = \sqrt{\lim_{x \to \infty} \dfrac{8x^2 + 3}{2x^2 + x}} = \sqrt{\lim_{x \to \infty} \dfrac{8 + \frac{1}{x^2}}{2 + \frac{1}{x}}} = \sqrt{\dfrac{8 + \lim_{x \to \infty} \frac{1}{x^2}}{2 + \lim_{x \to \infty} \frac{1}{x}}} =$

   $\sqrt{\dfrac{8 + 0}{2 + 0}} = \sqrt{\dfrac{8}{2}} = \sqrt{4} = \boxed{2}$.

3. (a) $f(x) = x^{\frac{3}{2}}(e^x + e^3)$.

   $$f'(x) = \frac{3}{2} x^{\frac{1}{2}}(e^x + e^3) + x^{\frac{3}{2}}(e^x + 0) = \boxed{\frac{3}{2} e^x x^{\frac{1}{2}} + \frac{3}{2} e^3 x^{\frac{1}{2}} + x^{\frac{3}{2}} e^x}.$$

   (b) $g(x) = \sqrt{\ln(2x + 1)} = \big[\ln(2x + 1)\big]^{\frac{1}{2}}$.

   $$g'(x) = \frac{1}{2} \big[\ln(2x + 1)\big]^{\frac{-1}{2}} . \frac{d}{dx} \ln(2x + 1) = \frac{1}{2} \frac{1}{\sqrt{\ln(2x + 1)} . \frac{1}{2x + 1}} .2 = \boxed{\frac{1}{(2x + 1)\sqrt{\ln(2x + 1)}}}.$$

4. (a) $\int \left[\dfrac{7}{1 + x^2} - \dfrac{5}{\sqrt{1 - x^2}}\right] dx = \boxed{7 \arctan(x) - 5 \arcsin(x) + C}$

   (b) $\int_0^{\frac{\pi}{3}} (1 + 2\sec\theta \tan\theta) d\theta = (\theta + 2\sec\theta)\Big|_0^{\frac{\pi}{3}} = (\frac{\pi}{3} + 2\sec(\frac{\pi}{3}) - (0 + 2\sec(0)) = \frac{\pi}{3} + 2(2) - 0 - 2 = \boxed{\frac{\pi}{3} + 2}$

5.

6. $y = \dfrac{(x + 1)^2}{1 + x^2} = \dfrac{x^2 + 2x + 1}{1 + x^2}$

(a) $\frac{dy}{dx} = \frac{(1+x^2)(x^2+2x+1)' - (x^2+2x+1)(1+x^2)}{(1+x^2)^2} = \frac{(1+x^2)(2x+2) - (x^2+2x+1)(2x)}{(1+x^2)^2} =$

$\frac{2x+2+2x^3+2x^2-2x^3-4x^2-2x}{(1+x^2)^2} = \boxed{\frac{2-2x^2}{(1+x^2)^2}}$.

(b) When $x = 0$, $y = \frac{(0+1)^2}{1+0} = \frac{1^2}{1} = 1$, $\boxed{y = 2}$

$m = \frac{2-0}{(1+0)^2} = 2$

The equation of the tangent Line:$y - y_1 = m(x - x_1)$

So, $y - 1 = 2(x - 0) \Longrightarrow \boxed{y = 2x + 1}$.

(c) The graph of the curve has horizontal tangents when $\frac{dy}{dx} = 0$

$\frac{2-2x^2}{(1+x^2)^2} = 0 \Longrightarrow 2 - 2x^2 = 0 \Longrightarrow 2(1-x^2) = 0 \Longrightarrow 2(1-x)(1+x) = 0 \Longrightarrow \boxed{x = -1, x = 1}$

7. $S = 2\pi r^2 + 2\pi rh$, $\frac{dr}{dt} = \frac{1}{2}$ in/sec, $\frac{dh}{dt} = -1$ in/sec,

Find $\frac{ds}{dt}$, when $r = 2$ in, $h = 3$ in.

$\frac{ds}{dt} = 2\pi(2r\frac{dr}{dt} + 2\pi h\frac{dr}{dt} + 2\pi r\frac{dh}{dt} = 4\pi(2)(\frac{1}{2}) + 2\pi(3)\frac{1}{2} + 2\pi(2)(-1) = 4\pi + 3\pi - 4\pi = 3\pi.$

$\boxed{\frac{ds}{dt} = 3\pi\ in^2/sec}$.

8.

9. $f(t) = 30e^{-0.23t}$

(a) Total change= $f(2) - f(0) = 30e^{-0.23(2)} - 30e^{-0.23(0)} = 30e^{-0.46} - 30 = \boxed{-11.06\ milligrams}$

(b) The Average rate of change $= \frac{f(2)-f(0)}{2-0} = \frac{-11.06}{2} = \boxed{-5.53\ milligrams}$

(c) $f'(t) = 30(-0.23)e^{-0.23t} = -6.9e^{-0.23t}$

The instantaneous rate of change $= f'(1) = -6.9e^{-0.23(1)} = \boxed{-5.48\ milligrams}$

10. $f(x) = \cos x - \sin^2 x$ on $[0, \pi]$.

$f'(x) = -\sin x - 2\sin x \cos x$

$f'(x) = 0 \Longrightarrow -\sin x - 2\sin x \cos x = 0 \Longrightarrow -\sin x(1 + 2\cos) = 0$

$\Longrightarrow \sin x = 0 \Longrightarrow x = 0, \pi.$

or

$\Longrightarrow 1 + 2\cos x = 0 \Longrightarrow \cos x = -\frac{1}{2} \Longrightarrow x = \frac{2\pi}{3}.$

$f(0) = \cos 0 - \sin^2 0 = 1$

$f(\pi) = \cos \pi - \sin^2 \pi = -1$

$f(\frac{2\pi}{3}) = \cos(\frac{2\pi}{3}) - \sin^2(\frac{2\pi}{3}) = -\frac{1}{2} - (\frac{\sqrt{3}}{2})^2 = -\frac{1}{2} - \frac{3}{4} = -\frac{5}{4}$

Absolute minimum $= \boxed{-\frac{5}{4}}$ Absolute maximum $= \boxed{1}$

11. $g(x) = \int_{-3}^{x} f(t)dt.$

(a) $g(5) = \int_{-3}^{5} f(t)dt = \frac{1}{2}\pi(2)^2 - \frac{1}{2}(4)(2) = 2\pi - 4.$

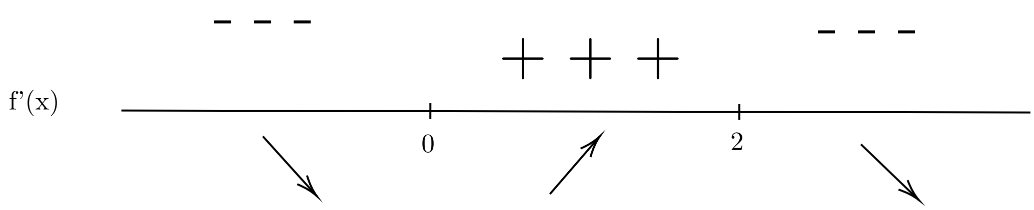
(b) At $x = 1$.

(c) Concave up on $(-3, -1)$.

(d) Inflection points at $x = -1$.

12. Sign chart for $f'(x)$:

$f'(x) = -\dfrac{1}{x(x-2)} \Longrightarrow x = 0, 2$ critical numbers

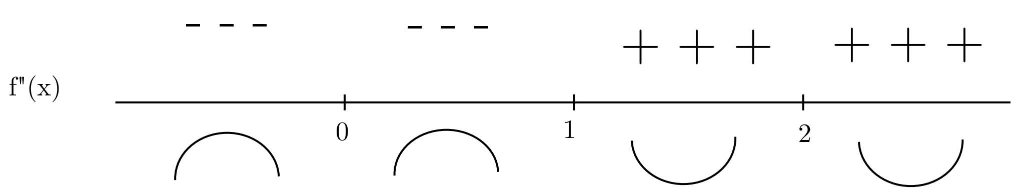


$f(x)$ is increasing on $(0, 2)$.

$f(x)$ is decreasing on $(-\infty, 0) \cup (2, \infty)$.

Sign chart for $f''(x)$:

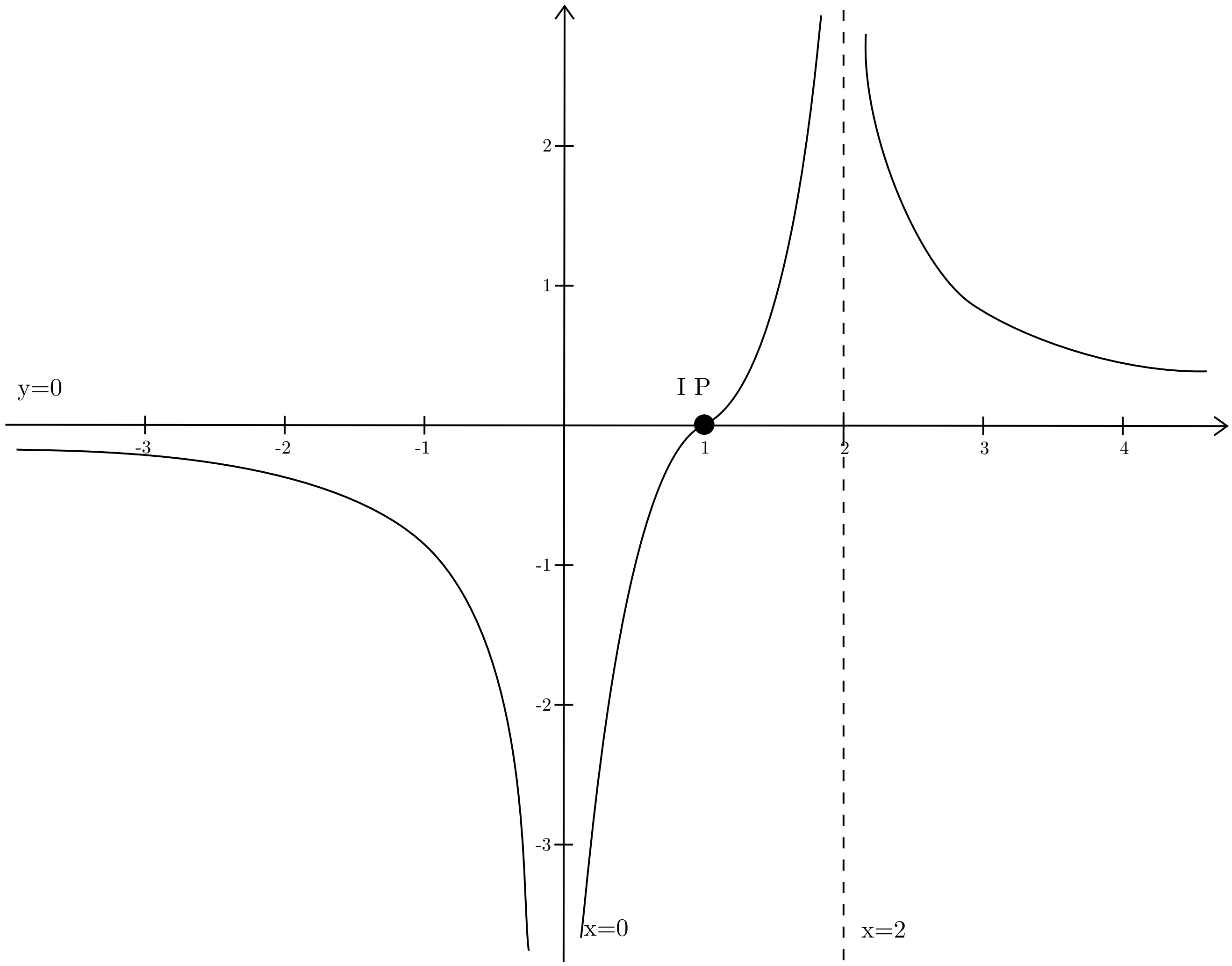
$f''(x) = \dfrac{2(x-1)}{x^2(x-2)} \Longrightarrow x = 0, 1, 2$ critical numbers



$f(x)$ is concave uo on $(1, 2) \cup (2, \infty)$

$f(x)$ is concave down on $(-\infty, 0) \cup (0, 1)$ $f(x)$ has an inflection point at $x = 1$

$f(x)$ has a vertical asymptote at $x = 0, x = 2$ and a Horizontal asymptote at $y = 0$.


y=0
I P
x=0
x=2
-3
-2
-1
1
2
3
4
2
1
-1
-2
-3