Problem 1

Set up the difference quotient:

$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{2-(x+h)}-\sqrt{2-x}}{h} = \frac{\sqrt{2-(x+h)}-\sqrt{2-x}}{h} \cdot \frac{\sqrt{2-x-h}+\sqrt{2-x}}{\sqrt{2-x-h}+\sqrt{2-x}}$$

$$= \frac{2-x-h-(2-x)}{h\left(\sqrt{2-x-h}+\sqrt{2-x}\right)} = \frac{-h}{h\left(\sqrt{2-x-h}+\sqrt{2-x}\right)} = \frac{-1}{\sqrt{2-x-h}+\sqrt{2-x}}$$

Using the definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{-1}{\sqrt{2-x-h}+\sqrt{2-x}} = \frac{-1}{\sqrt{2-x-0}+\sqrt{2-x}} = \frac{-1}{2\sqrt{2-x}}$$

Problem 2

Part(a): The limit does not exist since the directional limits do not agree:

$$\lim_{t\to 3^+} \frac{|t-3|}{t^2-2t-3} = \lim_{t\to 3^+} \frac{t-3}{(t-3)(t+1)} = \lim_{t\to 3^+} \frac{1}{(t+1)} = \frac{1}{4}$$

$$\lim_{t\to 3^-} \frac{|t-3|}{t^2-2t-3} = \lim_{t\to 3^+} \frac{-(t-3)}{(t-3)(t+1)} = \lim_{t\to 3^+} \frac{-1}{(t+1)} = \frac{-1}{4}$$

Part(b):

$$\lim_{x\to\infty} \frac{\sqrt{x^4+5x}}{2x-5} = \lim_{x\to\infty} \frac{\sqrt{x^2\left(x^2+\frac{5}{x}\right)}}{x\left(2-\frac{5}{x}\right)} = \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{\left(x^2+\frac{5}{x}\right)}}{x\left(2-\frac{5}{x}\right)} = \lim_{x\to\infty} \frac{|x|\sqrt{\left(x^2+\frac{5}{x}\right)}}{x\left(2-\frac{5}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{x\sqrt{\left(x^2+\frac{5}{x}\right)}}{x\left(2-\frac{5}{x}\right)} = \lim_{x\to\infty} \frac{\sqrt{\left(x^2+\frac{5}{x}\right)}}{2-\frac{5}{x}} = \frac{\sqrt{\infty}}{2} = \infty$$

Part(c): The limit does not exist since the left-sided limit is not defined:

$$\lim_{x\to 0^-} x^2\ln(x) = (0^-)^2\ln(0^-) \text{ Does Not Exist}$$

since $\ln(x)$ is not defined for negative values

Problem 3

Part(a):

$$f'(x) = \frac{(\cos(x))'\arctan(x)-\cos(x)(\arctan(x))'}{(\arctan(x))^2} = \frac{-\sin(x)\arctan(x)-\cos(x)\frac{1}{1+x^2}}{(\arctan(x))^2}$$

Part(b):

$$h(x) = 7[\sec(3x)]^6(\sec(3x))' = 7[\sec(3x)]^6\sec(3x)\tan(3x)(3x)' = 21[\sec(3x)]^6\sec(3x)\tan(3x)$$

Problem 4

Part(a):

$$\int \left[\sec^2(x) + \frac{1}{\sqrt{1-x^2}}\right] dx = \int \sec^2(x)dx + \int \frac{1}{\sqrt{1-x^2}}dx = \tan(x) + \arcsin(x) + C$$

Part(b):

$$\int_1^4 \left[\frac{1}{x} + \frac{1}{\sqrt{x}}\right] dx = \int_1^4 \frac{1}{x}dx + \int_1^4 \frac{1}{\sqrt{x}}dx = \ln|x|\Big|_1^4 + \int_1^4 x^{-\frac{1}{2}}dx$$

$$= \ln(4) - \ln(1) + 2x^{\frac{1}{2}}\Big|_1^4 = \ln(4) - 0 + 2(4)^{\frac{1}{2}} - 2(1)^{\frac{1}{2}} = \ln(4) + 4 - 2 = \ln(4) + 2$$

Problem 5

$$\frac{d}{dx}\left(\sin(xy) + \pi\right) = \frac{d}{dx}\left(e^{x-y}\right) \implies \cos(xy) \cdot \frac{d}{dx}(xy) = e^{x-y} \cdot \frac{d}{dx}(x-y)$$

$$\implies \cos(xy)\left(y + x\frac{dy}{dx}\right) = e^{x-y}\left(1 - \frac{dy}{dx}\right) \implies y\cos(xy) + x\cos(xy)\frac{dy}{dx} = e^{x-y} - e^{x-y}\frac{dy}{dx}$$

$$\implies x\cos(xy)\frac{dy}{dx} + e^{x-y}\frac{dy}{dx} = e^{x-y} - y\cos(xy) \implies \frac{dy}{dx}\left(x\cos(xy) + e^{x-y}\right) = e^{x-y} - y\cos(xy)$$

$$\implies \frac{dy}{dx} = \frac{e^{x-y} - y\cos(xy)}{x\cos(xy) + e^{x-y}}$$

Problem 6

Horizontal tangents correspond to points where the derivative is zero.

$$\frac{dy}{dx} = 3x^2 - 6x + 1 = 0 \implies x = \frac{-(-6) \pm \sqrt{36 - 4 \cdot 3 \cdot 1}}{2 \cdot 3} = \frac{6 \pm \sqrt{24}}{6} = \frac{3 \pm \sqrt{6}}{3}$$

To get the points, substitute the x-values back into the original function $f(x)$:

$$f\left(\frac{3+\sqrt{6}}{3}\right) = \left(\frac{3+\sqrt{6}}{3}\right)^3 - 3\left(\frac{3+\sqrt{6}}{3}\right)^2 + \left(\frac{3+\sqrt{6}}{3}\right) + 3 = \frac{81+33\sqrt{6}}{27} - 3 \cdot \frac{15+\sqrt{6}}{9} + \frac{3+\sqrt{6}}{3} + 3$$

$$= \frac{27+11\sqrt{6}}{9} + \frac{-45-27\sqrt{6}}{9} + \frac{9+3\sqrt{6}}{9} + \frac{27}{9} = \frac{18-13\sqrt{6}}{9}$$

$$f\left(\frac{3-\sqrt{6}}{3}\right) = \left(\frac{3-\sqrt{6}}{3}\right)^3 - 3\left(\frac{3-\sqrt{6}}{3}\right)^2 + \left(\frac{3-\sqrt{6}}{3}\right) + 3 = \frac{81-33\sqrt{6}}{27} - 3 \cdot \frac{15-\sqrt{6}}{9} + \frac{3-\sqrt{6}}{3} + 3$$

$$\frac{27-11\sqrt{6}}{9} - 3 \cdot \frac{15-6\sqrt{6}}{9} + \frac{3(3-\sqrt{6})}{9} + \frac{27}{9} = \frac{18+4\sqrt{6}}{9}$$

Problem 7

In order to take this derivative we need to rewrite the original function to take advantage of logarithm properties:

$$y = e^{\ln\left(\tan(x)^{x^3}\right)} = e^{x^3 \ln(\tan(x))}$$

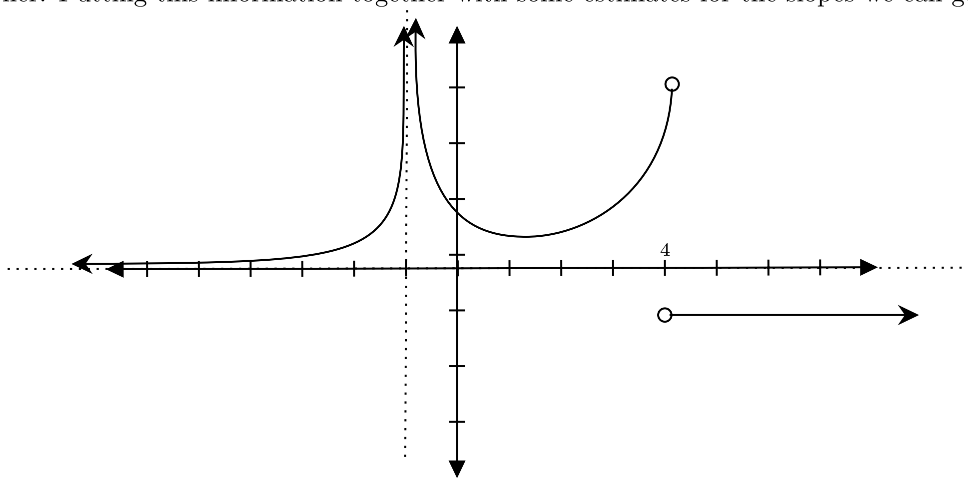
Take the derivative:

$$y' = \left(e^{x^3 \ln(\tan(x))}\right)' = e^{x^3 \ln(\tan(x))} \cdot \left(x^3 \ln(\tan(x))\right)' = e^{x^3 \ln(\tan(x))} \cdot \left(3x^2 \ln(\tan(x)) + x^3 \frac{1}{\tan(x)} \tan(x)'\right)$$

$$= e^{x^3 \ln(\tan(x))} \cdot \left(3x^2 \ln(\tan(x)) + x^3 \frac{\sec^2(x)}{\tan(x)}\right)$$

Problem 8

$f(x)$ has positive slopes on $(-\infty, -1) \cup (-1, 4)$ and negative slopes on $(4, \infty)$. The slope is not defined at $x = -1$ or at $x = 4$ since the graph's slope does not make sense at the asymptote and the corner. Putting this information together with some estimates for the slopes we can get a graph:



Problem 9

We want two numbers $x > 0$ and $y > 0$ such that $x + y = 120$ and $f(x) = x^2 y$ is at a maximum. Solving the first equation for y gives $y = 120 - x$. Substitute this into $f(x)$ and take it's derivative: $f'(x) = \left(x^2(120 - x)\right)' = \left(120x^2 - x^3\right)' = 240x - 3x^2$. The maximum will when $f'(x) = 0$, so $0 = 240x - 3x^2 = 3x(80 - x) \implies x = 0, 80$. Since we want x to be positive $(x > 0)$ then the solution is $x = 80$. Double check this is a maximum by plugging $x = 80$ in the second derivative: $f''(x) = 240 - 6x \implies f''(80) = -240$ meaning $f(x)$ is concave down at $x = 80$, so a maximum occurs. The final answer is $x = 80$ and $y = 40$.

Problem 10

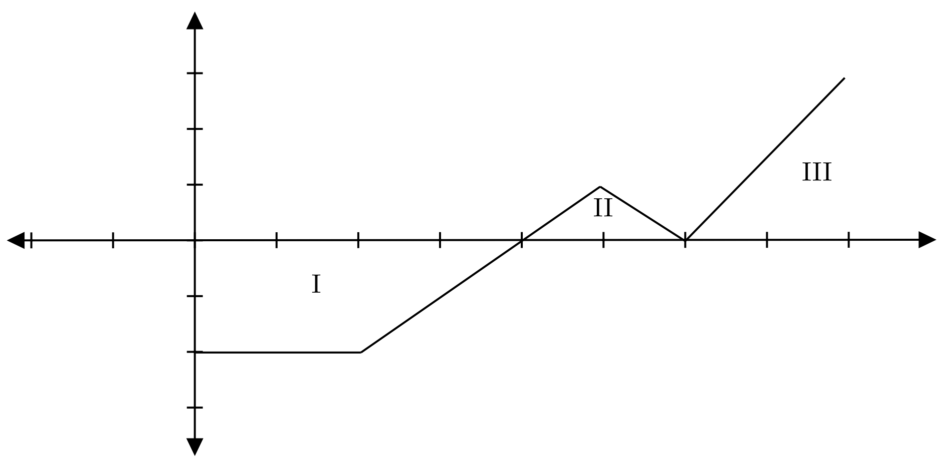
Since t is the number of hours past midnight, 6pm to midnight corresponds to $t = 6 + 12 = 18$ to $t = 24$.

$$\int_{18}^{24} 100 + 72t - 3t^2 dt = 100t + \frac{72t^2}{2} - \frac{3t^3}{3}\Big|_{18}^{24}$$
$$= 100(24) + 36(24)^2 - 23^3 - \left(100(18) + 36(18)^2 - 18^3\right) = 1690 \text{ gallons}$$

Problem 11

The function $g(x)$ gives area under the curve from 0 to x. So, $g(8)$ is the area under the curve from 0 to 8. This is the same as adding the areas of the polygons labeled I,II,III in the picture below with negative areas associated with regions below the x-axis.

$$g(8) = -A_I + A_{II} + A_{III} = -\frac{1}{2} \cdot (2)(2+4) + \frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 3$$

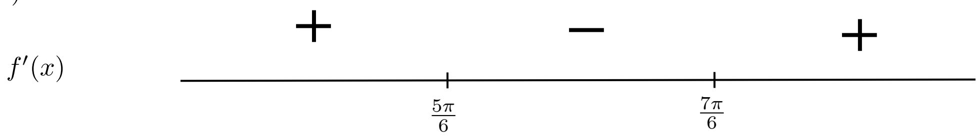


Problem 12

Sign chart for $f'(x)$:

$$\cos(x)+\frac{\sqrt{3}}{2}=0 \implies \cos(x)=-\frac{\sqrt{3}}{2} \implies x=\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) \implies x=\frac{5\pi}{6},\frac{7\pi}{6} \text{ critical numbers on } [0,2\pi]$$

$f'(x)$ has no "bad" numbers where it is undefined.

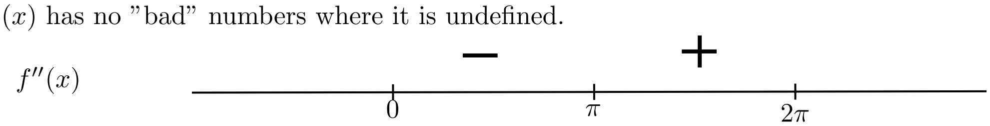


Between 0 and 2π, $f(x)$ is increasing on $(0,\frac{5\pi}{6})\cup(\frac{7\pi}{6},2\pi)$
Between 0 and 2π, $f(x)$ is decreasing on $(\frac{5\pi}{6},\frac{7\pi}{6})$
Between 0 and 2π, $f(x)$ has a local max $x=\frac{5\pi}{6}$
Between 0 and 2π, $f(x)$ has a local min $x=\frac{7\pi}{6}$

Sign chart for $f''(x)$:

$$-sin(x)=0 \implies x=0,\pi,2\pi \text{ critical numbers on } [0,2\pi]$$

$f''(x)$ has no "bad" numbers where it is undefined.



Between 0 and 2π, $f(x)$ is concave down on $(0,\pi)$
Between 0 and 2π, $f(x)$ is concave up on $(\pi,2\pi)$
Between 0 and 2π, $f(x)$ has one inflection point at $x=\pi$
Putting all of this information together we get the sketch: