MAT 2010, Fall 2019 Final Exam Solutions

1. (10 points) Use the **definition** of the derivative to differentiate the following function.

$$f(x) = \frac{x}{x+3}$$

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{x+h}{x+h+3} - \dfrac{x}{x+3}}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{(x+h)(x+3)}{(x+h+3)(x+3)} - \dfrac{x(x+h+3)}{(x+h+3)(x+3)}}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{(x+h)(x+3)-x(x+h+3)}{(x+h+3)(x+3)}}{h} \\
&= \lim_{h\to 0} \left(\frac{(x+h)(x+3)-x(x+h+3)}{(x+h+3)(x+3)} \cdot \frac{1}{h} \right) \\
&= \lim_{h\to 0} \frac{(x+h)(x+3)-x(x+h+3)}{h(x+h+3)(x+3)} \\
&= \lim_{h\to 0} \frac{x^2+3x+hx+3h-x^2-xh-3x}{h(x+h+3)(x+3)} \\
&= \lim_{h\to 0} \frac{3h}{h(x+h+3)(x+3)} \\
&= \lim_{h\to 0} \frac{3}{(x+h+3)(x+3)} \\
&= \frac{3}{(x+3)(x+3)} \\
&= \frac{3}{(x+3)^2}
\end{aligned}$$

$$\boxed{f'(x) = \frac{3}{(x+3)^2}}$$

2. (7 points each) Find the exact value of each of the following limits. Write "∞", "$-\infty$", or "does not exist" if appropriate. It is particularly important to show your work on this problem.

(a) $\displaystyle\lim_{x\to 3} \frac{x^3-9x}{x^2-2x-3}$

(b) $\lim_{x\to\infty} \dfrac{3 - 2x^2 + 5x^4}{2x^4 - 5}$

(c) $\lim_{x\to 0} \dfrac{\tan(\sqrt{x})}{\sqrt{x}}$

Solution:

(a) Note that, as x approaches 3, $\dfrac{x^3 - 9x}{x^2 - 2x - 3}$ approaches $\dfrac{0}{0}$, which is an indeterminate form. Therefore, more work is needed.

$$\begin{aligned}
&\lim_{x\to 3} \frac{x^3 - 9x}{x^2 - 2x - 3} \\
&= \lim_{x\to 3} \frac{x(x^2 - 9)}{(x-3)(x+1)} \\
&= \lim_{x\to 3} \frac{x(x-3)(x+3)}{(x-3)(x+1)} \\
&= \lim_{x\to 3} \frac{x(x+3)}{x+1} \\
&= \frac{18}{4} = \frac{9}{2}
\end{aligned}$$

$$\boxed{\lim_{x\to 3} \frac{x^3 - 9x}{x^2 - 2x - 3} = \frac{9}{2}}$$

(b) Note that, as x approaches ∞, $\dfrac{3 - 2x^2 + 5x^4}{2x^4 - 5}$ approaches $\dfrac{\infty}{\infty}$, which is an indeterminate form. Therefore, more work is needed.

$$\begin{aligned}
&\lim_{x\to\infty} \frac{3 - 2x^2 + 5x^4}{2x^4 - 5} \\
&= \lim_{x\to\infty} \frac{x^4\left(\frac{3}{x^4} - \frac{2}{x^2} + 5\right)}{x^4\left(2 - \frac{5}{x^4}\right)} && \text{(via force factoring)} \\
&= \lim_{x\to\infty} \frac{\frac{3}{x^4} - \frac{2}{x^2} + 5}{2 - \frac{5}{x^4}} \\
&= \frac{0 - 0 + 5}{2 - 0} = \frac{5}{2}
\end{aligned}$$

$$\boxed{\lim_{x\to\infty} \frac{3-2x^2+5x^4}{2x^4-5} = \frac{5}{2}}$$

(c) Note that, as x approaches 0, $\frac{\tan(\sqrt{x})}{\sqrt{x}}$ approaches $\frac{0}{0}$, which is an indeterminate form. Therefore, more work is needed.

$$\begin{aligned}
&\lim_{x\to 0} \frac{\tan(\sqrt{x})}{\sqrt{x}} \\
&= \lim_{x\to 0} \frac{\sec^2(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{x}}} && \text{(by L'Hospital's Rule)} \\
&= \lim_{x\to 0} \sec^2(\sqrt{x}) \\
&= \sec^2(0) \\
&= \frac{1}{\cos^2(0)} \\
&= \frac{1}{1^2} = 1
\end{aligned}$$

$$\boxed{\lim_{x\to 0} \frac{\tan(\sqrt{x})}{\sqrt{x}} = 1}$$

3. (7 points each) Differentiate the following functions. Simplify your answer.

(a) $f(x) = \dfrac{1+x^2}{\arctan(x)}$

(b) $g(x) = [\sec(3x)]^5$

Solution:

(a)

$$\begin{aligned}
f'(x) &= \frac{(1+x^2)'\arctan(x) - (\arctan(x))'(1+x^2)}{(\arctan(x))^2} && \text{(by the quotient rule)} \\
&= \frac{2x\arctan(x) - \frac{1}{1+x^2}(1+x^2)}{(\arctan(x))^2} \\
&= \frac{2x\arctan(x) - 1}{(\arctan(x))^2}
\end{aligned}$$

$$f'(x) = \frac{2x\arctan(x) - 1}{(\arctan(x))^2}$$

(b)

$$\begin{aligned} f'(x) &= 5[\sec(3x)]^4 \cdot (\sec(3x))' && \text{(by the chain rule)} \\ &= 5[\sec(3x)]^4 \cdot \sec(3x)\tan(3x) \cdot (3x)' && \text{(again by the chain rule)} \\ &= 5[\sec(3x)]^4 \cdot \sec(3x)\tan(3x) \cdot 3 \\ &= 15[\sec(3x)]^4 \cdot \sec(3x)\tan(3x) \\ &= 15[\sec(3x)]^5 \tan(3x) \end{aligned}$$

$$f'(x) = 15\sec^5(3x)\tan(3x)$$

4. Evaluate. Simplify your answer.

(a) (7 points) $\displaystyle\int \sec x\Big[\sec x + \tan x\Big]dx$

(b) (8 points) $\displaystyle\int_0^1 \Big[3\sqrt{t} - 2e^t\Big]\,dt$

Solution:

(a)

$$\begin{aligned} &\int \sec x\Big[\sec x + \tan x\Big]dx \\ &= \int \Big(\sec^2 x + \sec x\tan x\Big)dx \\ &= \tan x + \sec x + C \end{aligned}$$

$$\int \sec x\Big[\sec x + \tan x\Big]dx = \tan x + \sec x + C$$

(b)

$$\begin{aligned}
&\int_0^1 \left[3\sqrt{t} - 2e^t\right] dt \\
&= \int_0^1 \left[3t^{\frac{1}{2}} - 2e^t\right] dt \\
&= \left(3 \cdot \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 2e^t\right)\Bigg|_0^1 \\
&= \left(2t^{\frac{3}{2}} - 2e^t\right)\Big|_0^1 \\
&= (2-2e) - (0-2) = 4 - 2e
\end{aligned}$$

$$\boxed{\int_0^1 \left[3\sqrt{t} - 2e^t\right] dt = 4 - 2e}$$

5. (10 points) Consider the equation

$$y^4 + xy = x^3 - x + 2$$

(a) Find $\frac{dy}{dx}$.

(b) Find the equation of the tangent line at the point $(1, 1)$.

Solution:

(a) We use implicit differentiation here.

$$\begin{aligned}
y^4 + xy &= x^3 - x + 2 \\
\Rightarrow \frac{d}{dx}(y^4 + xy) &= \frac{d}{dx}(x^3 - x + 2) \\
\Rightarrow 4y^3\frac{dy}{dx} + \frac{d}{dx}(x)y + \frac{d}{dx}(y)x &= 3x^2 - 1 \\
\Rightarrow 4y^3\frac{dy}{dx} + 1y + \frac{dy}{dx}x &= 3x^2 - 1 \\
\Rightarrow 4y^3\frac{dy}{dx} + x\frac{dy}{dx} &= 3x^2 - y - 1 \\
\Rightarrow \frac{dy}{dx}(4y^3 + x) &= 3x^2 - y - 1 \\
\Rightarrow \frac{dy}{dx} &= \frac{3x^2 - y - 1}{4y^3 + x}
\end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{3x^2 - y - 1}{4y^3 + x}}$$

(b)

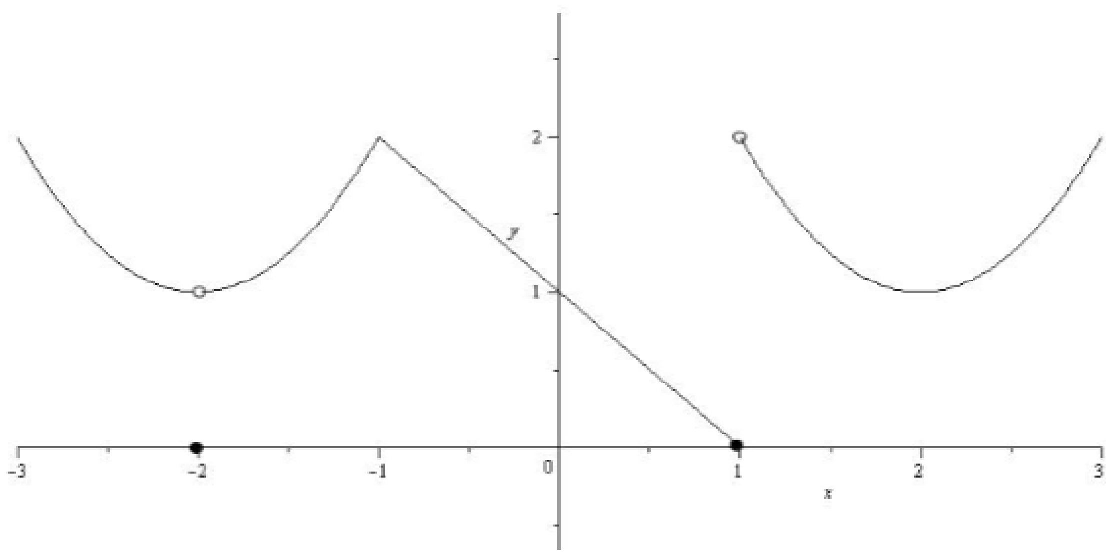
$$\begin{aligned} \left.\frac{dy}{dx}\right|_{(1,1)} &= \frac{3(1)^2 - 1 - 1}{4(1)^3 + 1} \\ &= \frac{1}{5} \end{aligned}$$

Hence, $\frac{1}{5}$ is the slope of the tangent line at the point $(1, 1)$. The equation of the tangent line at the point $(1, 1)$ has the form $y = \frac{1}{5}x + b$. We only need to solve for b now.

$$\begin{aligned} y &= \frac{1}{5}x + b \\ \Rightarrow 1 &= \frac{1}{5}(1) + b \\ b &= \frac{4}{5} \end{aligned}$$

$$\boxed{y = \frac{1}{5}x + \frac{4}{5}}$$

6. (10 points) The graph of a function $f(x)$ is given below.



(a) For which values of x in the interval $(-3, 3)$ is f not continuous? Give the name of each discontinuity.

(b) For which values of x in the interval $(-3, 3)$ is f not differentiable?

(c) Give values of (i) $\lim_{x \to 1^-} f(x)$ (ii) $\lim_{x \to 1^+} f(x)$ (iii) $\lim_{x \to -1} f(x)$

Solution:

(a) $f(-2) = 0$, but $\lim_{x \to -2} f(x) = 1$. Since $f(-2) \neq \lim_{x \to -2} f(x)$, there is a discontinuity at $x = -2$. This is a removable discontinuity.

$f(1) = 0$, but $\lim_{x \to 1} f(x)$ does not exist since $\lim_{x \to 1^-} f(x) = 0$ and $\lim_{x \to 1^+} f(x) = 2$. Hence there is a discontinuity at $x = 1$. This is a jump discontinuity.

$\boxed{x = -2 \text{ (removable discontinuity) and } x = 1 \text{ (jump discontinuity)}}$

(b) f is not continuous at $x = -2$ and $x = 1$, so f is not differentiable at $x = -2$ and $x = 1$. Also, f has a cusp/corner at $x = -1$, so f is not differentiable at $x = -1$.

$\boxed{x = -2, x = -1 \text{ and } x = 1}$

(c) $\boxed{\lim_{x \to 1^-} f(x) = 0, \lim_{x \to 1^+} f(x) = 2 \text{ and } \lim_{x \to -1} f(x) = 2}$

7. (10 points) On a typical day, a city consumes water at the rate of $r(t) = 100 + 72t - 3t^2$ gallons per hour, where t is the number of hours past midnight. How much water is consumed between 6 A.M. and 9 A.M.?

Solution:

$$\begin{aligned} &\int_6^9 r(t)dt \\ &= \int_6^9 (100 + 72t - 3t^2)dt \\ &= \left(100t + 72 \cdot \frac{t^2}{2} - 3 \cdot \frac{t^3}{3}\right)\Bigg|_6^9 \\ &= \left(100t + 36t^2 - t^3\right)\Bigg|_6^9 \\ &= (900 + 2916 - 729) - (600 + 1296 - 216) = 1407 \end{aligned}$$

$\boxed{1407 \text{ gallons of water are consumed between 6 A.M. and 9 A.M.}}$

8. (10 points) The position function of a spring in motion is given by

$$s(t) = 2e^{-1.5t}\sin(2\pi t),$$

where s is measured in centimeters and t in seconds.

(a) Find the velocity of the spring after t seconds.

(b) Find the velocity of the spring after 2 seconds. Give you answer correct to three decimal places. Include proper units.

Solution:

(a)

$$\begin{aligned} v(t) &= s'(t) \\ &= (2e^{-1.5t}\sin(2\pi t))' \\ &= 2(e^{-1.5t}\sin(2\pi t))' \\ &= 2((e^{-1.5t})'\sin(2\pi t) + e^{-1.5t}(\sin(2\pi t))') \\ &= 2(e^{-1.5t}\cdot(-1.5t)'\sin(2\pi t) + e^{-1.5t}\cos(2\pi t)\cdot(2\pi t)') \\ &= 2(e^{-1.5t}\cdot -1.5\sin(2\pi t) + e^{-1.5t}\cos(2\pi t)2\pi) \\ &= -3e^{-1.5t}\sin(2\pi t) + 4\pi e^{-1.5t}\cos(2\pi t) \end{aligned}$$

$$\boxed{v(t) = -3e^{-1.5t}\sin(2\pi t) + 4\pi e^{-1.5t}\cos(2\pi t)}$$

(b)

$$\begin{aligned} v(2) &= -3e^{-1.5\cdot 2}\sin(2\pi\cdot 2) + 4\pi e^{-1.5\cdot 2}\cos(2\pi\cdot 2) \\ &= -3e^{-3}\sin(4\pi) + 4\pi e^{-3}\cos(4\pi) \\ &= -3e^{-3}\cdot 0 + 4\pi e^{-3}\cdot 1 \\ &\approx 0.626 \end{aligned}$$

Approximately 0.626 cm/s

9. (10 points) The base of a triangle is decreasing at a rate of 2 ft/min and the height of the triangle is increasing at the rate of 1.5 ft/min. How fast is the area of the triangle changing when the base of the triangle is 5 ft long and the height is 3 ft?

Solution: Let A represent the triangle's area, b represent its base, and h represent its height. We know that $\frac{db}{dt} = -2$ ft/min, that $\frac{dh}{dt} = 1.5$ ft/min, and that $b = 5$ ft and $h = 3$ ft at the moment in question. We want $\frac{dA}{dt}$ at the moment in question.

$$
\begin{aligned}
A &= \frac{1}{2}bh \\
\Rightarrow \frac{d}{dt}(A) &= \frac{d}{dt}\Big(\frac{1}{2}bh\Big) \\
\Rightarrow \frac{dA}{dt} &= \frac{1}{2}\frac{d}{dt}(bh) \\
\Rightarrow \frac{dA}{dt} &= \frac{1}{2}\Big(\frac{d}{dt}(b)h + \frac{d}{dt}(h)b\Big) \\
\Rightarrow \frac{dA}{dt} &= \frac{1}{2}\Big(\frac{db}{dt}h + \frac{dh}{dt}b\Big)
\end{aligned}
$$

Then, at the moment in question,

$$
\begin{aligned}
\frac{dA}{dt} &= \frac{1}{2}(-2 \cdot 3 + 1.5 \cdot 5) \\
&= \frac{1}{2}(1.5) = 0.75
\end{aligned}
$$

The area of the triangle is changing at a rate of 0.75 ft^2/min

10. (10 points) Find the absolute minimum and absolute maximum values of $f(x) = x - \ln x$ on the interval $\left[\frac{1}{2}, 2\right]$.

Solution: On the interval $\left[\frac{1}{2}, 2\right]$, $f'(x) = 1 - \frac{1}{x}$ has a zero at $x = 1$ and no values of x at which it is undefined. Therefore, on the interval $\left[\frac{1}{2}, 2\right]$, $f(x)$ has only one critical number at $x = 1$. Substituting in this critical number and both endpoints of the interval for x in $f(x)$, we get $f\left(\frac{1}{2}\right) = \frac{1}{2} - \ln\left(\frac{1}{2}\right) \approx 1.19$, $f(1) = 1 - \ln(1) = 1$, and $f(2) = 2 - \ln(2) \approx 1.31$.

Absolute minimum value : 1, absolute maximum value : $2 - \ln(2)$

11. (10 points) Using a Riemann sum with $n = 4$ subintervals, find the overestimate (i.e. upper Riemann sum) of the area of the region bounded above by the function $f(x) = 2 + \sqrt{2x}$ and below by the x-axis on the interval $[0, 2]$.

Solution: Note that the function $f(x) = 2 + \sqrt{2x}$ is strictly increasing on the interval $[0, 2]$, so the upper Riemann sum coincides with the right Riemann sum. Then, we have $\Delta x = \dfrac{2-0}{4} = \dfrac{1}{2}$, $c_1 = \dfrac{1}{2}$, $c_2 = 1$, $c_3 = \dfrac{3}{2}$, and $c_4 = 2$. Therefore,

$$\begin{aligned}\text{Upper Riemann sum} &= \Delta x(f(c_1) + f(c_2) + f(c_3) + f(c_4)) \\ &= \frac{1}{2}\Big(f\Big(\frac{1}{2}\Big) + f(1) + f\Big(\frac{3}{2}\Big) + f(2)\Big) \\ &= \frac{1}{2}(3 + 2 + \sqrt{2} + 2 + \sqrt{3} + 4) \\ &= \frac{11}{2} + \frac{\sqrt{2}+\sqrt{3}}{2}\end{aligned}$$

$$\boxed{\frac{11}{2} + \frac{\sqrt{2}+\sqrt{3}}{2}\ \text{un.}^2}$$

12. (20 points) Sketch the graph of the function $f(x)$ which satisfies the following conditions. Use interval notation list all intervals where the function f is decreasing, increasing, concave up, and concave down. List the x-coordinates of all local maxima and minima, and points of inflection. Show asymptotes with dashed lines and give their equations. Label all important points on the graph.

(i) $f(x)$ is defined for all real numbers

(ii) $f'(x) = (x^2 - 2x - 3)e^x$

(iii) $f''(x) = (x^2 - 5)e^x$

(iv) $f(0) = 3$

(v) $\lim_{x\to\infty} f(x) = \infty$

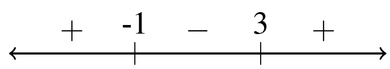
(vi) $\lim_{x\to-\infty} f(x) = 2$

Solution: The domain of f is $\mathbb{R}$ (as given in the first condition), f has a y-intercept at $(0, 3)$ (as given in the fourth condition), and f has a horizontal asymptote at $y = 2$ (which we deduce from the last condition).

Setting $f'(x)$ equal to zero, we have the following:

$$
\begin{aligned}
(x^2-2x-3)e^x &= 0\\
\Rightarrow x^2-2x-3=0 \text{ or } &\underbrace{e^x=0}_{\text{No solution}}\\
\Rightarrow (x-3)(x+1) &= 0\\
x=3, x &= -1
\end{aligned}
$$

$f'(x)$ has no values of x at which it is undefined. Therefore, f has two critical numbers at $x=3$ and $x=-1$. The corresponding sign chart of $f'(x)$ is below, where each of -1 and 3 are zeroes.

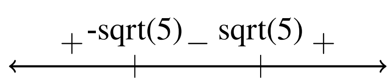


With this information, we see that f is increasing on $(-\infty,-1)\cup(3,\infty)$ and is decreasing on $(-1,3)$. We also see that f has a local maximum at $x=-1$ and a local minimum at $x=3$.

Setting $f''(x)$ equal to zero, we have the following:

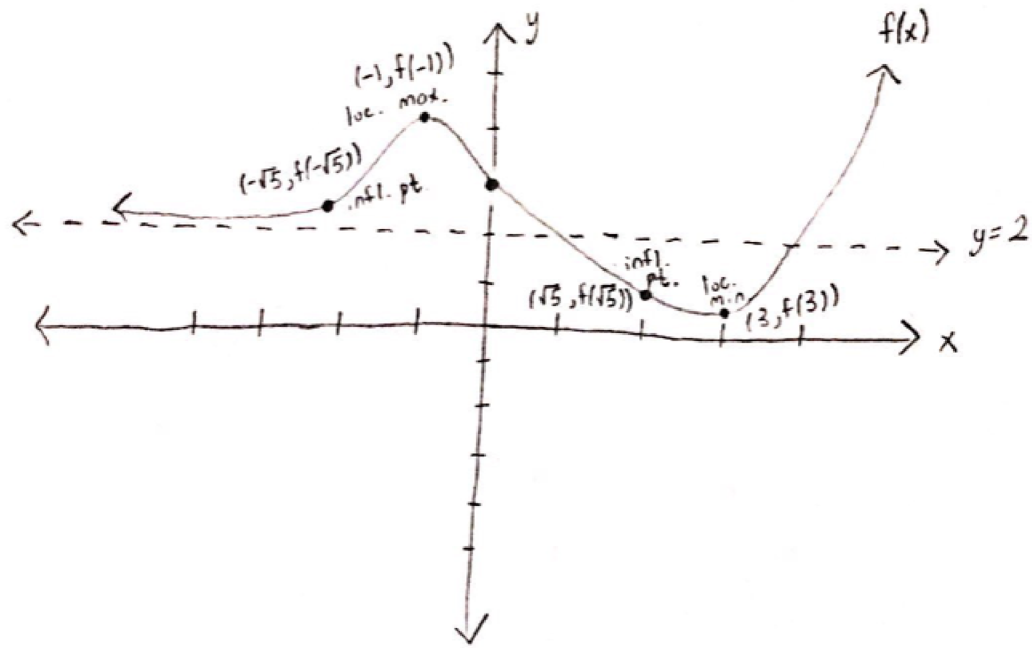
$$
\begin{aligned}
(x^2-5)e^x &= 0\\
\Rightarrow x^2-5=0 \text{ or } &\underbrace{e^x=0}_{\text{No solution}}\\
\Rightarrow x &= \pm\sqrt{5}
\end{aligned}
$$

$f''(x)$ has no values of x at which it is undefined. The corresponding sign chart of $f''(x)$ is below, where each of $-\sqrt{5}$ and $\sqrt{5}$ are zeroes.



With this information, we see that f is concave up on $(-\infty,-\sqrt{5})\cup(\sqrt{5},\infty)$ and is concave down on $(-\sqrt{5},\sqrt{5})$. We also see that f has inflection points at $x=-\sqrt{5}$ and $x=\sqrt{5}$.

The desired graph of $f(x)$ is shown below.



y
f(x)
(-1, f(-1))
loc. max.
(-√5, f(-√5))
infl. pt
y=2
infl. pt.
(√5, f(√5))
loc. min
(3, f(3))
x