MAT 1800, Winter 2020 Final Exam Solutions

1. If $f(x) = x^2 + x + 1$ and $g(x) = x - 4$, find and simplify:

a) $f(x) - g(x)$

b) $(f \circ g)(x)$

c) $(g \circ f)(3)$

Solution:

a)

$$\begin{aligned} &f(x) - g(x) \\ &= x^2 + x + 1 - (x - 4) \\ &= x^2 + x + 1 - x + 4 \\ &= x^2 + 5 \end{aligned}$$

$\boxed{f(x) - g(x) = x^2 + 5}$

b)

$$\begin{aligned} &(f \circ g)(x) \\ &= f(g(x)) \\ &= f(x - 4) \\ &= (x - 4)^2 + (x - 4) + 1 \\ &= (x - 4)(x - 4) + x - 4 + 1 \\ &= x^2 - 4x - 4x + 16 + x - 4 + 1 \\ &= x^2 - 7x + 13 \end{aligned}$$

$\boxed{(f \circ g)(x) = x^2 - 7x + 13}$

c) Note that $f(3) = 3^2 + 3 + 1 = 9 + 3 + 1 = 13$. Therefore,

$$\begin{aligned} &(g \circ f)(3) \\ &= g(f(3)) \\ &= g(13) \\ &= 13 - 4 = 9 \end{aligned}$$

$\boxed{(g \circ f)(3) = 9}$

2. Let $f(x) = \dfrac{x-5}{3x+2}$. Find $f^{-1}(x)$, where $f^{-1}(x)$ is the inverse of the function of f.

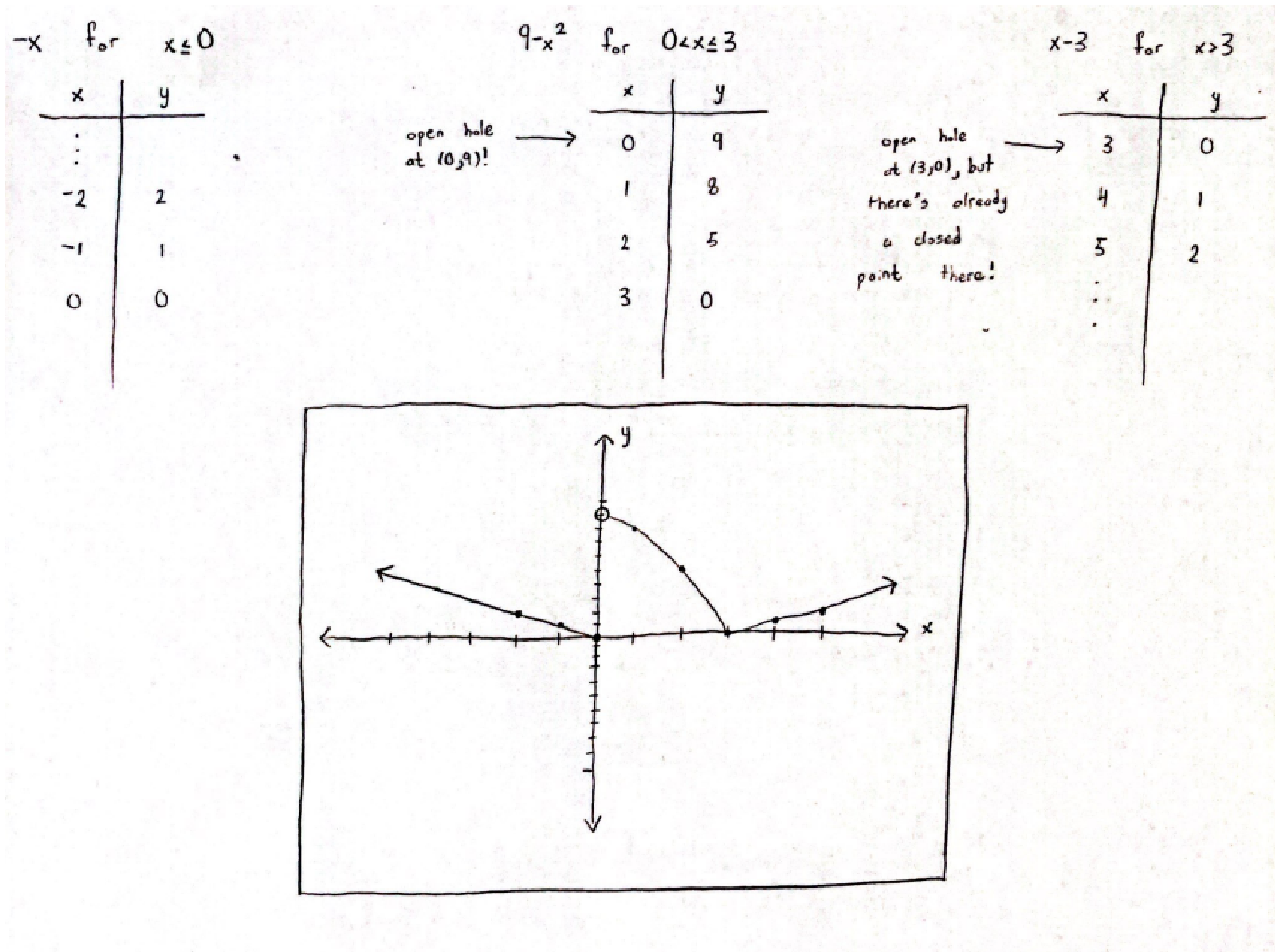
Solution: We are given the function $y = \dfrac{x-5}{3x+2}$. In order to find its inverse function, we can solve this equation for x, as follows:

$$
\begin{aligned}
y &= \frac{x-5}{3x+2} \\
\Rightarrow y(3x+2) &= \frac{x-5}{3x+2}(3x+2) \\
\Rightarrow y(3x+2) &= x-5 \\
\Rightarrow 3xy+2y &= x-5 \\
\Rightarrow 3xy &= x-2y-5 \\
\Rightarrow 3xy-x &= -2y-5 \\
\Rightarrow x(3y-1) &= -2y-5 \\
\Rightarrow x &= \frac{-2y-5}{3y-1}
\end{aligned}
$$

$$\boxed{f^{-1}(x) = \frac{-2x-5}{3x-1}}$$

3. Sketch a graph of the function $f(x) = \begin{cases} -x & x \le 0 \\ 9-x^2 & 0 < x \le 3 \\ x-3 & x > 3 \end{cases}$

Solution: Before graphing our piecewise function $f(x)$, we create a table of values for each of the three "pieces" of f here. We make note where we have a table entry that will correspond to an open hole on our graph. To more accurately draw our graph, we also note that the first piece of our graph is part of a line with slope -1, the second is part of a parabola opening down and the third is part of a line with slope 1.



4. A right triangle has one leg that is four times as long as the other leg. Find a function that models the triangle's perimeter, P, in terms of x, the length of the shorter leg.

Solution: Let y be the length of the longer leg of the triangle and let z be the length of the hypotenuse of the triangle. Then, $P = x + y + z$. Our goal is to get P in terms of the single variable x. Since the longer leg is four times as long as the shorter leg, we have $y = 4x$, giving $P = x + 4x + z$, or $P = 5x + z$.

In order to get P in terms of only x, we still need to solve for z in terms of x. For this, note that we have a right triangle, so we can use the Pythagorean Theorem as follows:

$$
\begin{aligned}
x^2 + y^2 &= z^2 \\
\Rightarrow x^2 + (4x)^2 &= z^2 \\
\Rightarrow x^2 + 16x^2 &= z^2 \\
\Rightarrow 17x^2 &= z^2 \\
\Rightarrow \sqrt{17x^2} &= \sqrt{z^2} \\
\Rightarrow \sqrt{17}x &= z
\end{aligned}
$$

Note that, in the final step, we only took the positive square root. This is because z is a length and hence should be positive.

Therefore, $P = 5x + z$ implies $P = 5x + \sqrt{17}x$, or, after factoring out an x, $P = x(5 + \sqrt{17})$.

$\boxed{P(x) = x(5 + \sqrt{17})}$

5. Find the domain of the following functions:

a) $f(x) = \sqrt{7x - 3}$

b) $h(x) = \log(x^2 - 16)$

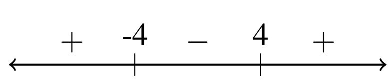
Solution:

a) There is only one domain restriction at play here, which is $7x - 3 \geq 0$. Since this is a linear inequality, we can solve it as follows:

$$
\begin{aligned}
7x - 3 &\geq 0 \\
\Rightarrow 7x &\geq 3 \\
\Rightarrow x &\geq \frac{3}{7}
\end{aligned}
$$

$\boxed{\left[\frac{3}{7}, \infty\right)}$

b) There is only one domain restriction at play here, which is $x^2 - 16 > 0$. Since this is a nonlinear inequality, we need a sign chart for the function $g(x) = x^2 - 16 = (x - 4)(x + 4)$. This is shown below, where each of -4 and 4 are zeroes.



Noting the positive regions, we see that our desired domain is $(-\infty, -4) \cup (4, \infty)$.

$$\boxed{(-\infty, -4) \cup (4, \infty)}$$

6. Find the average rate of change of the function $f(t) = \dfrac{3}{t}$ from $t = a$ to $t = a + h$ and simplify your answer so that no single factor of h is left in the denominator.

Solution:

$$\begin{aligned}
&\frac{f(a+h) - f(a)}{a+h-a} \\
&= \frac{\dfrac{3}{a+h} - \dfrac{3}{a}}{h} \\
&= \frac{\dfrac{3}{a+h} \cdot \dfrac{a}{a} - \dfrac{3}{a} \cdot \dfrac{a+h}{a+h}}{h} \\
&= \frac{\dfrac{3a}{a(a+h)} - \dfrac{3(a+h)}{a(a+h)}}{h} \\
&= \frac{\dfrac{3a - 3(a+h)}{a(a+h)}}{h} \\
&= \frac{3a - 3(a+h)}{a(a+h)} \cdot \frac{1}{h} \\
&= \frac{3a - 3(a+h)}{ha(a+h)} \\
&= \frac{3a - 3a - 3h}{ha(a+h)} \\
&= \frac{-3h}{ha(a+h)} \\
&= \frac{-3}{a(a+h)}
\end{aligned}$$

$$\boxed{\frac{-3}{a(a+h)}}$$

7. A ball is thrown straight up in the air. The height, in feet, of the ball t seconds after being thrown is given by the function $h(t) = -3t^2 + 12t + 6$.

a) What is the maximum height reached by the ball?

b) How many seconds after being thrown does the ball reach this height?

Solution:

(a) The graph of $h(t)$ is a parabola opening down. The x-coordinate of the vertex of the parabola is given by $\frac{-12}{2(-3)} = \frac{-12}{-6} = 2$. This represents the number of seconds after being thrown that the ball reaches its maximum height. The y-coordinate of the vertex of the parabola is then given by $h(2) = -3(2)^2 + 12(2) + 6 = -3(4) + 24 + 6 = -12 + 24 + 6 = 18$. This represents the maximum height reached by the ball.

$\boxed{\text{18 feet}}$

(b) $\boxed{\text{2 seconds}}$

8. Find all zeros of the polynomial $p(x) = x^3 - 3x^2 - 8x - 10$. Express any non-real zeros in the form $a + bi$.

Solution: We need to be able to factor the polynomial $p(x) = x^3 - 3x^2 - 8x - 10$. By the Rational Roots Theorem, the possible rational zeros of $p(x)$ include $\pm 1, \pm 2, \pm 5$ and ± 10. Starting with 1, we can plug in values from this list for x in $p(x)$ until we find a zero of $p(x)$. We find that 5 is a zero of $p(x)$, since $p(5) = 5^3 - 3(5)^2 - 8(5) - 10 = 125 - 3(25) - 40 - 10 = 125 - 75 - 40 - 10 = 0$.

Since 5 is a zero of $p(x)$, we know that $x - 5$ is a factor of $p(x)$. Now, we can use polynomial long division to our advantage, as shown below.

$$
\begin{array}{r|llll}
 & x^2 & +2x & +2 & \\
\hline
x-5 & x^3 & -3x^2 & -8x & -10 \\
 & -x^3 & +5x^2 & & \\
\hline
 & & 2x^2 & -8x & \\
 & & -2x^2 & +10x & \\
\hline
 & & & 2x & -10 \\
 & & & -2x & +10 \\
\hline
 & & & & 0
\end{array}
$$

Hence we have $p(x) = x^3 - 3x^2 - 8x - 10 = (x - 5)(x^2 + 2x + 2)$. Then,

$$\begin{aligned} p(x) &= 0 \\ \Rightarrow (x-5)(x^2+2x+2) &= 0 \\ \Rightarrow x = 5 \text{ or } x^2+2x+2 &= 0 \\ \Rightarrow x = 5 \text{ or } x &= \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} \\ \Rightarrow x = 5 \text{ or } x &= \frac{-2 \pm \sqrt{-4}}{2} \\ \Rightarrow x = 5 \text{ or } x &= \frac{-2 \pm 2i}{2} \\ \Rightarrow x = 5 \text{ or } x &= -1 \pm i \end{aligned}$$

$\boxed{x = 5, -1-i, -1+i}$

9. Let $f(x) = \dfrac{x^2 - 2x - 8}{x^2 + 5x}$

a) Graph $f(x)$, labeling all intercepts and asymptotes.

b) State the domain and range of $f(x)$.

Solution:

a) First, we set the denominator of $f(x)$ equal to zero to solve for the vertical asymptotes:

$$\begin{aligned} x^2 + 5x &= 0 \\ \Rightarrow x(x+5) &= 0 \\ \Rightarrow x = 0 \text{ or } x + 5 &= 0 \\ \Rightarrow x = 0 \text{ or } x &= -5 \end{aligned}$$

Hence we have two vertical asymptotes at $x = 0$ and $x = -5$.

Next, since the degree of the denominator of $f(x)$ (which is 2) is equal to the degree of the numerator of $f(x)$, we consider the ratio of leading coefficients of the numerator and denominator, which is $\dfrac{1}{1} = 1$. Therefore, we have a horizontal asymptote at $y = 1$.

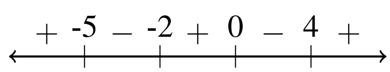
Let's now solve for the zeroes of $f(x)$ below:

$$
\begin{aligned}
f(x) &= 0 \\
\Rightarrow \frac{x^2-2x-8}{x^2+5x} &= 0 \\
\Rightarrow \frac{x^2-2x-8}{x^2+5x}\cdot(x^2+5x) &= 0\cdot(x^2+5x) \\
\Rightarrow x^2-2x-8 &= 0 \\
\Rightarrow (x-4)(x+2) &= 0 \\
\Rightarrow x-4=0 \ \text{ or } \ x+2 &= 0 \\
\Rightarrow x=4 \ \text{ or } \ x &= -2
\end{aligned}
$$

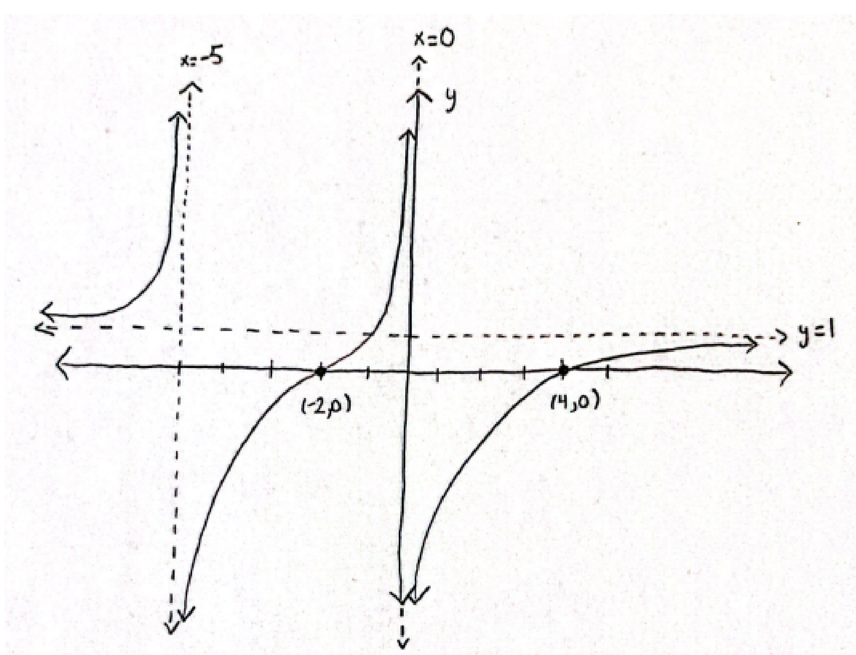
Hence $f(x)$ has two zeroes at $x = 4$ and $x = -2$. This also means that $f(x)$ has two x-intercepts at $(-2, 0)$ and $(4, 0)$.

To find the y-intercept of $f(x)$, we find $f(0)$. However, note that $f(0)$ is undefined, since the denominator of $f(x)$ is equal to 0 when $x = 0$. This means that $f(x)$ has no y-intercept.

Lastly, we create a sign chart for $f(x)$ below, with key points at the vertical asymptotes and zeroes from earlier.



The final graph is shown below.



b) We can read off both the domain and range of $f(x)$ from our graph of $f(x)$ above.

$\boxed{\text{Domain:}(-\infty,-5)\cup(-5,0)\cup(0,\infty), \text{Range:}(-\infty,\infty)}$

10. Find the exact value of each expression.

a) $\ln(e^6)$

b) $\log_3(\sqrt{27})$

c) $\log_2(80) - \log_2(5)$

Solution:

a) $\boxed{\ln\left(e^6\right) = 6}$

b)

$$\begin{aligned} &\log_3(\sqrt{27}) \\ &= \log_3(27^{\frac{1}{2}}) \\ &= \frac{1}{2}\log_3(27) \\ &= \frac{1}{2}\cdot 3 \\ &= \frac{3}{2} \end{aligned}$$

$\boxed{\log_3(\sqrt{27}) = \frac{3}{2}}$

c)

$$\begin{aligned} &\log_2(80) - \log_2(5) \\ &= \log_2\left(\frac{80}{5}\right) \\ &= \log_2(16) \\ &= 4 \end{aligned}$$

$\boxed{\log_2(80) - \log_2(5) = 4}$

11. The half-life of Strontium-90 is 28 years. How long will it take a 40mg sample to decay to a mass of 16mg?

Solution: We use the exponential decay function $Q(t) = Q_0 e^{rt}$ here, where $Q(t)$ is the amount (in mg) of Strontium-90 after t years, Q_0 is the initial amount of Strontium-90, and r is the decay rate. Since the half-life of Strontium-90 is 28 years, it takes 28 years for Q_0 to decay to $\frac{1}{2}Q_0$. That is,

$$\begin{aligned} Q(28) &= \frac{1}{2}Q_0 \\ \Rightarrow \frac{1}{2}Q_0 &= Q_0 e^{28r} \\ \Rightarrow \frac{1}{2} &= e^{28r} \\ \Rightarrow \ln\left(\frac{1}{2}\right) &= \ln\left(e^{28r}\right) \\ \Rightarrow \ln\left(\frac{1}{2}\right) &= 28r \\ \Rightarrow r &= \frac{\ln\left(\frac{1}{2}\right)}{28} \end{aligned}$$

We want to know how long it will take for a 40mg sample to decay to a mass of 16mg. Taking $Q_0 = 40$, we want to find the value of t for which $Q(t) = 16$. The work for this is shown below.

$$\begin{aligned} Q(t) &= 16 \\ \Rightarrow 16 &= 40e^{\frac{\ln\left(\frac{1}{2}\right)}{28}t} \\ \Rightarrow \frac{16}{40} &= e^{\frac{\ln\left(\frac{1}{2}\right)}{28}t} \\ \Rightarrow \frac{2}{5} &= e^{\frac{\ln\left(\frac{1}{2}\right)}{28}t} \\ \Rightarrow \ln\left(\frac{2}{5}\right) &= \ln\left(e^{\frac{\ln\left(\frac{1}{2}\right)}{28}t}\right) \\ \Rightarrow \ln\left(\frac{2}{5}\right) &= \frac{\ln\left(\frac{1}{2}\right)}{28}t \\ \Rightarrow \frac{28}{\ln\left(\frac{1}{2}\right)} \cdot \ln\left(\frac{2}{5}\right) &= \frac{28}{\ln\left(\frac{1}{2}\right)} \cdot \frac{\ln\left(\frac{1}{2}\right)}{28}t \\ \Rightarrow t &= \frac{28\ln\left(\frac{2}{5}\right)}{\ln\left(\frac{1}{2}\right)} \end{aligned}$$

$$\boxed{\frac{28\ln\left(\frac{2}{5}\right)}{\ln\left(\frac{1}{2}\right)} \text{ years}}$$

12. Find all values of x, if any, such that $\log_8(x+2) + \log_8(3) = \log_8(9) + \log_8(2x-11)$.

Solution:

$$\begin{aligned} \log_8(x+2) + \log_8(3) &= \log_8(9) + \log_8(2x-11) \\ \Rightarrow \log_8(3(x+2)) &= \log_8(9(2x-11)) \\ \Rightarrow \log_8(3x+6) &= \log_8(18x-99) \\ \Rightarrow 3x+6 &= 18x-99 \\ \Rightarrow 6 &= 15x-99 \\ \Rightarrow 105 &= 15x \\ \Rightarrow x &= 7 \end{aligned}$$

Checking our answer by plugging it back into the original equation, we don't run into the potential domain issue of taking the logarithm of a negative number. Therefore, $x = 7$ is a legitimate solution.

$$\boxed{x = 7}$$

13. Find the exact value of each.

a) $\sec\left(\frac{7\pi}{4}\right)$

b) $\cos\left(\frac{-2\pi}{3}\right)$

c) $\tan\left(\frac{-7\pi}{6}\right)$

Solution:

a)

$$\begin{aligned} &\sec\left(\frac{7\pi}{4}\right) \\ &= \frac{1}{\cos\left(\frac{7\pi}{4}\right)} \\ &= \frac{1}{\frac{\sqrt{2}}{2}} \\ &= \frac{2}{\sqrt{2}} \\ &= \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{2\sqrt{2}}{2} \\ &= \sqrt{2} \end{aligned}$$

$$\boxed{\sec\left(\frac{7\pi}{4}\right) = \sqrt{2}}$$

b)

$$\begin{aligned} &\cos\left(\frac{-2\pi}{3}\right) \\ &= \cos\left(\frac{-2\pi}{3} + 2\pi\right) \\ &= \cos\left(\frac{-2\pi}{3} + \frac{6\pi}{3}\right) \\ &= \cos\left(\frac{4\pi}{3}\right) \\ &= -\frac{1}{2} \end{aligned}$$

$$\boxed{\cos\left(\frac{-2\pi}{3}\right) = -\frac{1}{2}}$$

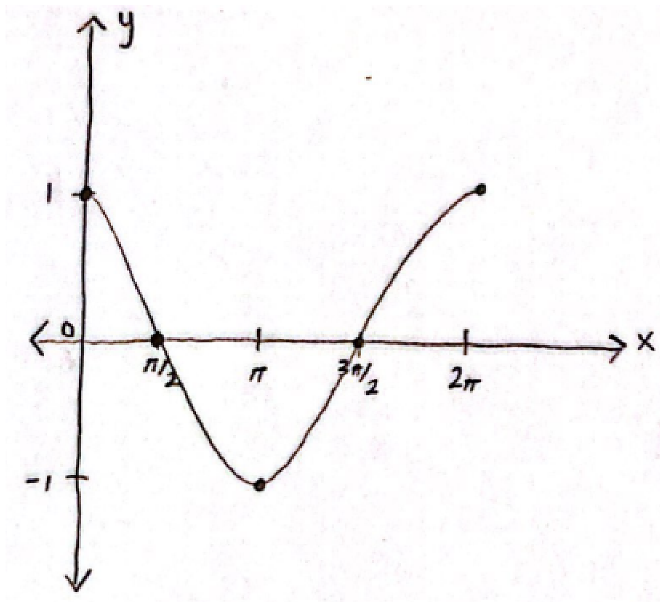
c)

$$\begin{aligned}
&\tan\left(\frac{-7\pi}{6}\right)\\
&= \tan\left(\frac{-7\pi}{6} + 2\pi\right)\\
&= \tan\left(\frac{-7\pi}{6} + \frac{12\pi}{6}\right)\\
&= \tan\left(\frac{5\pi}{6}\right)\\
&= \frac{\sin\left(\frac{5\pi}{6}\right)}{\cos\left(\frac{5\pi}{6}\right)}\\
&= \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}}\\
&= \frac{1}{2} \cdot -\frac{2}{\sqrt{3}}\\
&= \frac{-2}{2\sqrt{3}}\\
&= \frac{-1}{\sqrt{3}}\\
&= \frac{-1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}\\
&= -\frac{\sqrt{3}}{3}
\end{aligned}$$

$$\boxed{\tan\left(\frac{-7\pi}{6}\right) = -\frac{\sqrt{3}}{3}}$$

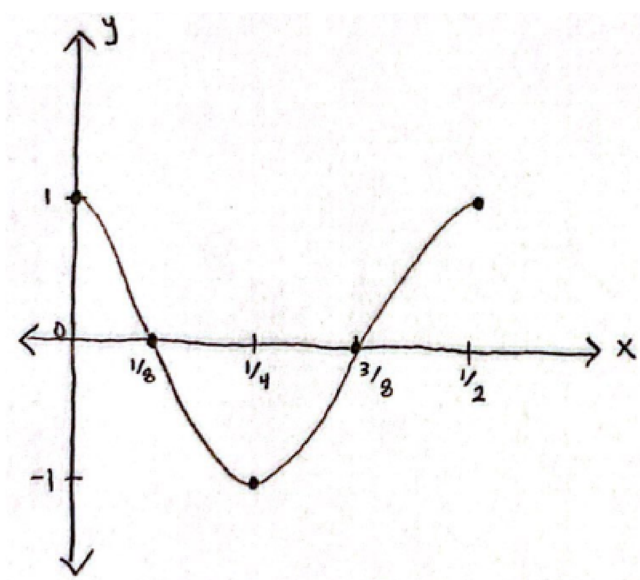
14. Graph the function $f(x) = \cos(4\pi x) - 2$ over one complete period. Show each transformation and label all high and low points.

Solution: We begin with the graph of $\cos(x)$ over one complete period, as shown below.

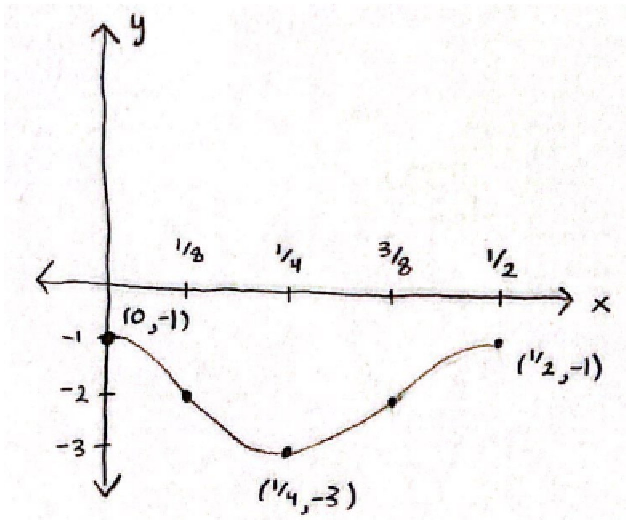


Note that $\cos(x)$ has period 2π, and $\dfrac{2\pi}{4} = \dfrac{\pi}{2}$ gives the length of each of the four subintervals in one complete period.

Next, we get the graph of $\cos(4\pi x)$. To do this, we note that the factor of 4π now in front of x is responsible for a change in the period; the period of $\cos(4\pi x)$ is $\dfrac{2\pi}{4\pi} = \dfrac{1}{2}$. The length of each of the four subintervals in one complete period is then $\dfrac{\frac{1}{2}}{4} = \dfrac{1}{8}$. A graph of $\cos(4\pi x)$ over one complete period is shown below.



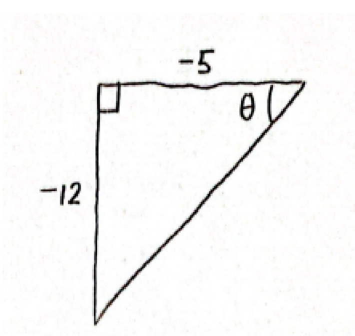
Lastly, to obtain the graph of $f(x) = \cos(4\pi x) - 2$, we simply take the graph of $\cos(4\pi x)$ and shift it down 2 units. Our final graph of $f(x)$ over one complete period is shown below.



15. Given that $\tan(\theta) = \dfrac{12}{5}$, θ is in Quadrant III, and $\sin(\alpha) = \dfrac{-\sqrt{10}}{10}$, α is in Quadrant IV, find and simplify $\sin(\theta - \alpha)$.

Solution: By use of the formula $\sin(x - y) = \sin(x)\cos(y) - \sin(y)\cos(x)$, we have $\sin(\theta - \alpha) = \sin(\theta)\cos(\alpha) - \sin(\alpha)\cos(\theta) = \sin(\theta)\cos(\alpha) + \dfrac{\sqrt{10}}{10}\cos(\theta)$. Hence we still need to find $\sin(\theta)$, $\cos(\alpha)$ and $\cos(\theta)$.

We know that $\tan(\theta) = \dfrac{12}{5}$ and θ is in Quadrant III. We now draw the appropriate right triangle with θ in Quadrant III and $\tan(\theta) = \dfrac{12}{5}$ below, keeping in mind that $\tan(\theta)$ represents the side length opposite from θ divided by the side length adjacent to θ.

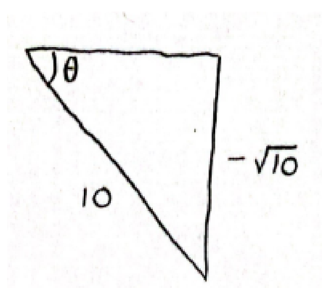


In order to determine both $\sin(\theta)$, which represents the side length opposite from θ divided by the hypotenuse length, and $\cos(\theta)$, which represents the side length adjacent to θ divided by the hypotenuse length, we need to use the Pythagorean Theorem to find the hypotenuse length c. This work is shown below.

$$\begin{aligned}(-12)^2 + (-5)^2 &= c^2 \\ \Rightarrow 144 + 25 &= c^2 \\ \Rightarrow 169 &= c^2 \\ \Rightarrow c &= 13\end{aligned}$$

Note that we've only taken the positive square root in the final step; the hypotenuse length is always taken to be positive in these types of problems. With this, we see that $\cos(\theta) = \dfrac{-5}{13}$ and $\sin(\theta) = \dfrac{-12}{13}$.

Next, we find $\cos(\alpha)$. We draw the appropriate right triangle with α in Quadrant IV and $\sin(\alpha) = \dfrac{-\sqrt{10}}{10}$ below.



In order to determine $\cos(\alpha)$, we need to use the Pythagorean Theorem to find the adjacent side b. This work is shown below.

$$\begin{aligned}(-\sqrt{10})^2 + b^2 &= 10^2 \\ \Rightarrow 10 + b^2 &= 100 \\ \Rightarrow b^2 &= 90 \\ \Rightarrow b = \sqrt{90} = \sqrt{9 \cdot 10} = \sqrt{9} \cdot \sqrt{10} &= 3\sqrt{10}\end{aligned}$$

Note that we've only taken the positive square root in the final step, since b will be positive based on θ being in Quadrant IV. With this, we see that $\cos(\alpha) = \dfrac{3\sqrt{10}}{10}$.

In all, we have the following:

$$
\begin{aligned}
&\sin(\theta-\alpha)\\
&=\sin(\theta)\cos(\alpha)+\frac{\sqrt{10}}{10}\cos(\theta)\\
&=\frac{-12}{13}\cdot\frac{3\sqrt{10}}{10}+\frac{\sqrt{10}}{10}\cdot\frac{-5}{13}\\
&=\frac{-36\sqrt{10}}{130}+\frac{-5\sqrt{10}}{130}\\
&=\frac{-36\sqrt{10}-5\sqrt{10}}{130}\\
&=\frac{-41\sqrt{10}}{130}
\end{aligned}
$$

$$\boxed{\sin(\theta-\alpha)=\frac{-41\sqrt{10}}{130}}$$

16. Find all primary solutions ($0 \le \theta < 2\pi$) of $\tan^3(\theta) = \tan(\theta)$.

Solution:

$$
\begin{aligned}
&\tan^3(\theta)=\tan(\theta)\\
&\Rightarrow \tan^3(\theta)-\tan(\theta)=0\\
&\Rightarrow \tan(\theta)(\tan^2(\theta)-1)=0\\
&\Rightarrow \tan(\theta)=0 \text{ or } \tan^2(\theta)-1=0\\
&\Rightarrow \tan(\theta)=0 \text{ or } \tan^2(\theta)=1\\
&\Rightarrow \tan(\theta)=0 \text{ or } \sqrt{\tan^2(\theta)}=\sqrt{1}\\
&\Rightarrow \tan(\theta)=0 \text{ or } \tan(\theta)=1 \text{ or } \tan(\theta)=-1\\
&\Rightarrow \theta=0,\pi \text{ or } \theta=\frac{\pi}{4},\frac{5\pi}{4} \text{ or } \theta=\frac{3\pi}{4},\frac{7\pi}{4}
\end{aligned}
$$

$$\boxed{\theta=0,\frac{\pi}{4},\frac{3\pi}{4},\pi,\frac{5\pi}{4},\frac{7\pi}{4}}$$