Ph.D. QUALIFYING EXAMINATION
DEPARTMENT OF PHYSICS AND ASTRONOMY
WAYNE STATE UNIVERSITY

PART I

Friday, May 6, 2022
10:00 – 12:00 pm

INSTRUCTIONS: This examination consists of three problems each worth 10 points. Use a separate booklet for each problem. Write the following information on the front cover of each booklet:

1. the **special ID number** that you received from Delores Cowen,
2. the **problem number** (*i.e.* Problem 7).

Please make sure your answers are dark and legible.

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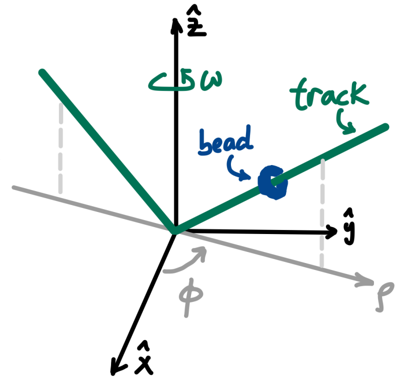
Problem 1 (10 points)

A bead of mass, m, is constrained to a frictionless "V-shaped" track, and this track rotates about the z-axis, as shown in the figure. The equation of the track is $z = k\rho$, where ρ is the distance perpendicular to the z-axis and k is a positive constant. The V-shaped track rotates about $\hat{\boldsymbol{z}}$ with constant angular velocity, ω.



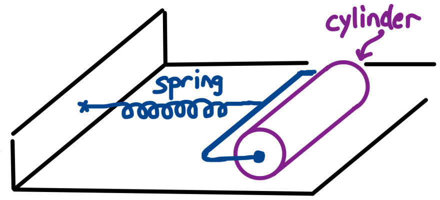
(a) [4 points] Taking the perpendicular distance from the z-axis, ρ, as the variable of interest, write the equation of motion for ρ.

(b) [2 points] Are there any equilibrium points?

(c) [4 points] Show that the equation of motion from part (a) reduces to the equation of motion for a frictionless inclined plane if the track is not being rotated.

Problem 2 (10 points)

A solid cylinder (radius R, mass M with uniform density ρ, length L, volume $\pi R^2 L$) is attached via a spring (massless, spring constant k) to a wall as shown in the figure. The horizontal surface is rough enough that the cylinder can only roll (no slipping).



(a) [4 points] Perform the integral $I = \int r^2 dm$ over the cylinder to show that the moment of inertia of the cylinder for rotations about its symmetry axis is $I = \frac{1}{2}MR^2$.

(b) [4 points] Letting the generalized coordinate be x, the position of the center of the cylinder with respect to its equilibrium position, what is the equation of motion?

(c) [2 points] What is the angular frequency, ω, for small oscillations about the equilibrium position?

Problem 3 (10 points)

A block of mass m is projected upward along an inclined plane that makes an angle θ with the horizontal plane as shown in the figure below. At the block-incline interface both the coefficients of static and kinetic friction have the same value μ. The initial speed of the block at the bottom of the incline is v_0.

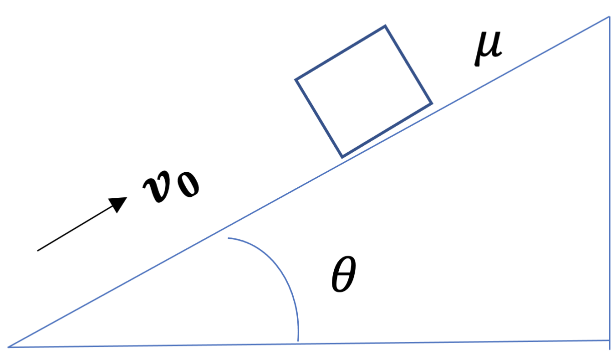
(a) [2 points] Determine, in terms of given quantities, the maximum distance D that the block moves up along the inclined plane and the time, t_{up}, it takes to reach that highest point.

(b) [2 points] Next, the block slides down the ramp to its starting point. What is the time of descent, t_{down}, of the block to the bottom of the ramp?

(c) [2 points] What is the largest value of the angle θ, in terms of given quantities, such that the block stops and stays at its highest point?

(d) [2 points] Now imagine that the block and the inclined plane are frictionless (that is, μ=0). What is the total up-and-down travel time for the block in this frictionless case?

(e) [2 points] What is the relationship between μ and θ if the total travel times are the same for the frictional and the frictionless motions?



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PART II

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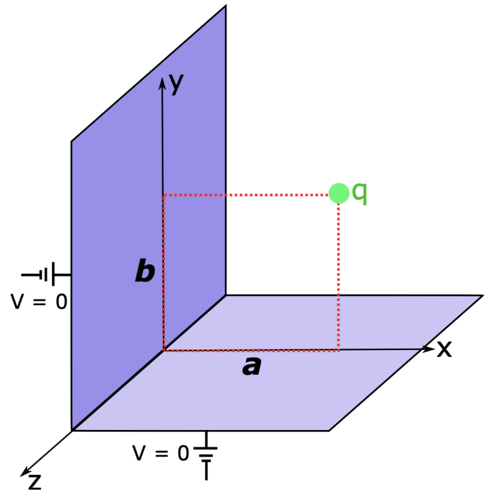
Problem 4 (10 points)



Two semi-infinite conducting sheets are grounded and connected at a right angle as seen in the figure. One sheet lies in the $\hat{\boldsymbol{x}}\hat{\boldsymbol{z}}$ plane and the other sheet lies in the $\hat{\boldsymbol{y}}\hat{\boldsymbol{z}}$ plane. A point charge, q, is placed at the coordinate $(x, y, z) = (a, b, 0)$.

(a) [2 points] Determine the locations of the image charges required to calculate the electric potential, $V(x, y, z)$, for $x, y > 0$.

(b) [3 points] Determine the electric potential, $V(x, y, z)$, for $x, y > 0$.

(c) [3 points] Calculate the work required to bring the charge to $(x, y, z) = (a, b, 0)$ from a point that is infinitely far away from the sheets.

(d) [2 points] Calculate the charge densities, $\sigma(x, z)$ and $\sigma(y, z)$, that are induced on the sheet in the $\hat{\boldsymbol{x}}\hat{\boldsymbol{z}}$ plane and on the sheet in the $\hat{\boldsymbol{y}}\hat{\boldsymbol{z}}$ plane.

Problem 5 (10 points)

A spherically symmetric charge distribution exists within a sphere of radius R. The electric field inside the sphere points radially outward, and is of the form $\mathbf{E} = [Ar + Br^2]\hat{\mathbf{r}}$.

(a) [2 points] Calculate the charge density, $\rho(r)$.

(b) [2 points] Calculate the electric field outside the sphere ($r > R$).

(c) [2 points] Calculate the electric potential, $V(r)$, for a point outside the sphere ($r > R$).

(d) [2 points] Calculate the electric potential, $V(r)$, for a point inside the sphere ($r < R$).

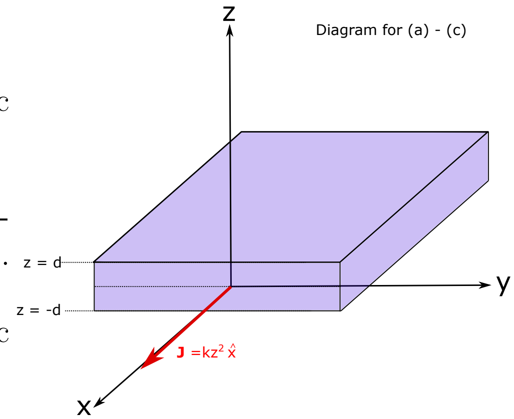
(e) [2 points] How much work is required to assemble the charge distribution?

Hint: In spherical coordinates the divergence of a vector function can be expressed as:

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta \sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}.$$

Problem 6 (10 points)

An infinite conducting slab is centered about the $\hat{\boldsymbol{x}}\hat{\boldsymbol{y}}$-plane and is shown in the upper figure to the right. The slab has a total thickness of $2d$. A current density $\mathbf{J} = (kz^2)\hat{\boldsymbol{x}}$ flows throughout the slab.

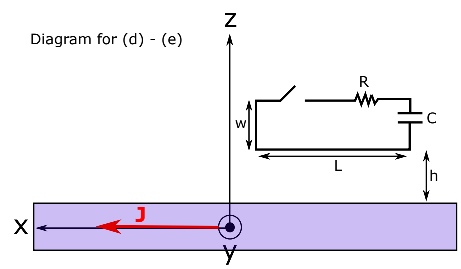


(a) [1 point] Determine the direction that the magnetic field points for $z < 0$ and $z > 0$.

(b) [3 points] Determine the magnitude of the magnetic field, $\mathbf{B}(z)$, at some point inside the slab ($|z| < d$).

(c) [3 points] Determine the magnitude of the magnetic field, $\mathbf{B}(z)$, at some point outside the slab ($|z| > d$).

A loop of wire with resistance R and capacitance C is placed a height h above the top surface of the conducting slab. As shown in the lower figure on the right, this loop is in the $\hat{\boldsymbol{z}}\hat{\boldsymbol{x}}$-plane. The loop has a length L and width w. At $t = 0$, a switch in the *loop* is closed and a complete circuit is formed. Simultaneously, the current density in the *slab* begins to increase linearly in time and is given by $\mathbf{J} = (ktz^2/T)\hat{\boldsymbol{x}}$, where T is a time constant.



(d) [2 points] Determine the direction of current flow in the loop of wire.

(e) [1 point] Calculate the current induced in the loop of wire as a function of time.

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PART III

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Problem 7 (10 points)

A system has three energy levels at $\epsilon=0$, $\epsilon=k_BT_a$, and $\epsilon=k_BT_b$, where k_B is the Boltzmann constant, T_a=300 K and T_b=600 K. The degeneracies of the levels are 1, 3, and 5, respectively. To receive full credit, provide *numerical* answers for all parts.

(a) [3 points] Calculate the single-particle partition function at a temperature of 300K.

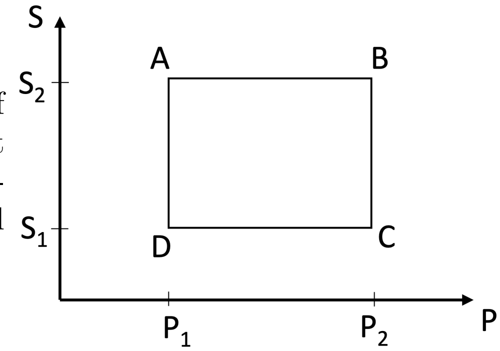
(b) [2 points] Calculate the relative populations of the energy levels at 300K.

(c) [2 points] Calculate the average energy per particle at 300K.

(d) [3 points] At what temperature is the population of the energy level at k_BT_b equal to the population of the energy level at k_BT_a?

Problem 8 (10 points)

A heat engine runs in Joule's cycle, which consists of two constant pressure (P) processes and two constant entropy (S) processes, as shown in the diagram. Assume that the working material is a monatomic ideal gas.



(a) [2 points] Make a qualitative drawing of the cycle in the pressure-volume (P-V) diagram. Label the isobaric (constant pressure) steps. Label the isentropic (constant entropy) steps.

(b) [1 point] If the engine is to be run to produce work, will the cycle be clockwise or counter-clockwise in the P-V diagram?

(c) [1 point] Which step in the P-V diagram has the heat coming into the engine? Give a reason.

(d) [1 point] Which step in the P-V diagram has the heat going out of the engine? Give a reason.

(e) [5 points] What is the efficiency of this heat engine in terms of the variables P_1 and P_2 only?

The following information might be useful.
For a constant-entropy process, PV^γ = constant, $TV^{\gamma-1}$ = constant, and $TP^{(1-\gamma)/\gamma}$ = constant, with $\gamma = 5/3$ for monatomic ideal gas.

Problem 9 (10 points)

A particle of mass m_0 at rest can decay into two particles of rest masses m_1 and m_2 only if the initial mass m_0 is greater than the sum of final masses, *i.e.* if the mass excess $\Delta = m_0 - m_1 - m_2$ is positive.

(a) [3 points] Derive the relativistic expressions for the kinetic energies of the two particles formed in the decay process with the mass m_0 initially at rest.

(b) [2 points] Show that the relativistic expressions for the kinetic energies of the two particles can be written as,

$$T_i = \left[1 - \frac{m_i}{m_0} - \frac{\Delta}{2m_0}\right] \Delta c^2, \qquad (i = 1, 2).$$

(c) [2 points] Verify explicitly that the sum of the kinetic energies of the two particles equals Δc^2.

(d) [3 points] A charged π meson of rest energy 139.6 MeV decays into a μ meson of rest energy 105.7 MeV and a neutrino with zero rest mass. Calculate the kinetic energies of the μ meson and the neutrino.

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Problem 10 (10 points)

A particle of mass m is confined to move freely inside a two-dimensional square box, with impenetrable sides each of length L. The sides are parallel to the $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ axes with $0 \leq x \leq L$ and $0 \leq y \leq L$.

(a) [4 points] Derive the energy eigenvalues and the corresponding eigenfunctions of this particle.

(b) [3 points] What is the degeneracy of the lowest two allowed energy states?

(c) [3 points] Now, a small perturbation $V = \beta xy$ is introduced, where β is a constant. Calculate the energy shift of the ground state through first order perturbation theory.

Problem 11 (10 points)

Consider the single-particle state of an electron.

(a) [5 points] For a corresponding spin operator $\hat{S}_x + \hat{S}_y + \hat{S}_z$, what are its possible eigenvalues? What is the normalized eigenvector (also called eigenspinor) corresponding to the smallest eigenvalue?

(b) [5 points] Initially the electron is in an eigenstate corresponding to the smallest eigenvalue of the above spin operator given in (a). What is the probability that a measurement of $\hat{S}_x$ gives the outcome of $+\hbar/2$? (Hint: first calculate the corresponding eigenvector of $\hat{S}_x$.)

Note the Pauli matrices are written as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Problem 12 (10 points)

A quantum rotator is governed by the Hamiltonian

$$\hat{H} = \frac{\hat{L}_x^2 + \hat{L}_y^2}{2I_1} + \frac{\hat{L}_z^2}{2I_2},$$

where L_x, L_y, and L_z are the three components of orbital angular momentum, and $I_1 \neq I_2$ are the moments of inertia and are constants.

(a) [3 points] What are the energy eigenvalues of this rotator?

(b) [1 point] What are the corresponding normalized eigenfunctions (obtained without any new calculations)? You need to briefly explain your answer.

(c) [6 points] When this rotator is subjected to a perturbation $\hat{H}' = E\sin(2\theta)$, where E is a small constant, calculate the first-order energy corrections for the states with the azimuthal quantum number $l = 1$.

Note the following spherical harmonics:

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2}, \quad Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta, \quad Y_1^{\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}.$$